Warm-Up: Discuss "Guidelines for Good Mathematical Writing."

Which suggestions do you think are most important?

Propositional Calculus

← Nothing to do with differential/integral calculus

How to "build" new sentences from existing ones?

Use logical connectives.

Logical Connective	Symbol	Plain English
negation	$\neg$	"not"
conjunction	$\wedge$	"and"
disjunction	$\vee$	"or" (inclusive)
implication	$\Longrightarrow$	"if-then"
biconditional	$\Longleftrightarrow$	"if and only if"

Let $P, Q, R, \ldots$ stand for sentences.

Ex: P = "It is Friday."
Q = "We're having fun in Math 3345."

$\neg P$, $P \wedge Q$, $P \Rightarrow Q$, etc.

① Negation: $\neg$ means "not"

The negation $\neg P$ has the opposite truth value as P.
So if P is true, then $\neg P$ is false,
if P is false, then $\neg P$ is true.

Summarize this in a truth table:

P	$\neg P$
T	F
F	T

Ex: What is $\neg(\neg P)$? Make another truth table:

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

If P is true, then $\neg P$ is false. If $\neg P$ is false, then $\neg(\neg P)$ is true. So if P is true then $\neg(\neg P)$ is true.

Similarly, if P is false, then $\neg(\neg P)$ is false.

So P and $\neg(\neg P)$ always have the same truth value. We say they are <u>logically equivalent</u> and write

$$P \equiv \neg(\neg P).$$

"is logically equivalent to"

② <u>Conjunction</u>: $\wedge$ means "and"

$P \wedge Q$ is true exactly when <u>both</u> P and Q are true.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex: 2 is even and 3 is odd. T
2 is even and 3 is even. F
2 is odd and 3 is odd. F

Observation: • $\wedge$ is commutative: $P \wedge Q \equiv Q \wedge P$

• $\wedge$ is associative: $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$

Proof by words/truth table (if time).

③ Disjunction: $\vee$ means "or" (inclusive)

$P \wedge Q$ is true exactly when at least one of P or Q is true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex: 2 is even or 3 is odd. T

2 is even or 3 is even. T

2 is odd or 3 is even. F

Note: $P \vee Q \equiv Q \vee P$

$$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$$

How do the operations $\neg$, $\wedge$, $\vee$ interact with one another?

Ex: Let

P = "m is even."

Q = "n is odd."

Then

$P \wedge Q$ is "m is even and n is odd."

This becomes false if $\underbrace{\text{m is not even } \underline{\text{OR}} \text{ n is not odd.}}_{= \neg P \vee \neg Q}$

Thm (DeMorgan's Laws) Let P and Q be sentences. Then

(a) $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$

(b) $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$

Proof of (a):

By truth table:

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

So we see $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$.

In words:

We wish to show $\neg(P \wedge Q)$ always has the same truth value as $\neg P \vee \neg Q$.

First, suppose $\neg(P \wedge Q)$ is true. Then $P \wedge Q$ is false, so at least one of P or Q is false.

But this means at least one of $\neg P$ or $\neg Q$ is true, so $\neg P \vee \neg Q$ is true.

Next, suppose $\neg(P \wedge Q)$ is false. Then $P \wedge Q$ is true, so both P and Q are true.

Now, both $\neg P$ and $\neg Q$ will be false, meaning $\neg P \vee \neg Q$ is false as well.

(b) HW 1