More about Sets

If A is a set, then $x \in A$ means x is an element of A. $x \notin A$ means x is not an element of A.

Ex: $A = \{1, 2, 3\}$. Then $2 \in A$ and ☺ $\notin A$.

Def: The empty set is the set with no elements. It is denoted $\emptyset$.

$x \in \emptyset$ is false for every x.

Sets can have sets as elements.

Ex: $A = \{\{1,2\}, \{\text{red}, \text{blue}\}, \$\}$ is a set with three elements, two of which are sets themselves.

$$\{1,2\} \in A$$

$$1 \notin A$$

Other ways to specify sets

In words:

- Let B be the set whose elements are the first five prime numbers

  so $B = \{2, 3, 5, 7, 11\}$

- Let $\mathbb{R}_{>0}$ be the set of all positive real numbers.

By patterns:

- $E = \{2, 4, 6, 8, \ldots\}$

  (E is the set of positive even numbers)

- $P = \{2, 3, 5, 7, 11, \ldots\}$

  (P is the set of all prime numbers)

These first two methods are somewhat limited.

Set-Builder Notation: If $P(x)$ is a sentence, then

$$\{x \mid P(x)\} \quad \text{or} \quad \{x : P(x)\}$$

"such that" "such that"

is the set of all x such that $P(x)$ is true.

If A is a set, then

$$\{x \in A \mid P(x)\}$$

is the set of all x such that $x \in A$ and $P(x)$ is true. (x is a bound variable)

- $E = \{n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\}$.

  Also, $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$.

  Also, $E = \{n \in \mathbb{N} : 2 \mid n\}$.

- $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$.

  Also, $\mathbb{R}_{>0} = \{y \in \mathbb{R} \mid y > 0\}$

Transformation notation: If A is a set and f is some function defined on A, then

$$\{f(x) \mid x \in A\}$$

is the set of all objects $f(x)$ obtained from all $x \in A$.

- $E = \{2n \mid n \in \mathbb{N}\}$
- $S = \{n^2 \mid n \in \mathbb{N}\}$ Transformation
  $= \{m \mid \text{there exists } n \in \mathbb{N} \text{ such that } n^2 = m\}$
  Set-Builder

Ex: Let $S = \{n^2 \mid n \in \mathbb{N}\}$ be the set of (positive) squares and $C = \{n^3 \mid n \in \mathbb{N}\}$ the set of (positive) cubes.

Define $A = \{x + y \mid x \in S \text{ and } y \in C\}$.

Then

<u>In words</u>: A is the set of integers which can be written as the sum of a positive square and a positive cube.

<u>Pattern</u>: $A = \{2, 5, 9, 10, 12, 17, 24, \ldots\}$

<u>Set-Builder</u>:

$A = \{n \in \mathbb{N} \mid \text{there exist } a, b \in \mathbb{N} \text{ such that } n = a^2 + b^3\}$

Subsets

Def: Let A and B be sets. We say A is a <u>subset</u> of B, written $A \subseteq B$, if $x \in A$ implies $x \in B$.

That is, $A \subseteq B$ means every element of A is also an element of B.

Can write as $(\forall x \in A)(x \in B)$
or $(\forall x)[x \in A \Rightarrow x \in B]$

Ex:
- $\{2,3,5\} \subseteq \{1,2,3,4,5\}$
- $\mathbb{N} \subseteq \mathbb{Z}$
- $\mathbb{Z} \subseteq \mathbb{Q}$
- $\mathbb{Q} \subseteq \mathbb{R}$

Ex: Any time we use set-builder notation to write
$$A = \{x \in B \mid P(x)\},$$
we have $A \subseteq B$.

Thm: For every set A, $\emptyset \subseteq A$.

Proof: The sentence $x \in \emptyset$ is always false.
Thus,

$$x \in \emptyset \Rightarrow x \in A$$

is always true, so $\emptyset \subseteq A$. ∎

Thm: For every set A, $A \subseteq A$.

Proof: The sentence

$$x \in A \Rightarrow x \in A$$

is true for all x, so $A \subseteq A$. ∎

Def: Let A and B be sets. We say $\underline{A = B}$ if $A \subseteq B$ and $B \subseteq A$.

So to prove $A = B$, we usually have to prove 2 things:

- $A \subseteq B$ $\quad (x \in A \Rightarrow x \in B)$
- $B \subseteq A$ $\quad (x \in B \Rightarrow x \in A)$

Thm: Let A and B be sets. Then $A = B$ if and only if $(x \in A \Leftrightarrow x \in B)$ for all x.

Proof: $A = B$ is logically equivalent to

$$(A \subseteq B) \wedge (B \subseteq A)$$

$$\equiv (\forall x)(x \in A \Rightarrow x \in B) \wedge (\forall x)(x \in B \Rightarrow x \in A)$$

$$\equiv (\forall x)\left[(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)\right]$$

$$\equiv (\forall x)\left[x \in A \Leftrightarrow x \in B\right].$$

since $\left[(\forall x) P(x)\right] \wedge \left[(\forall x) Q(x)\right] \equiv (\forall x)\left[P(x) \wedge Q(x)\right]$

$\blacksquare$

Ex: Let's prove

$$\underbrace{\{x \in \mathbb{Z} \mid x^2 = 1\}}_{=A} = \underbrace{\{1, -1\}}_{=B}$$

We must prove $A \subseteq B$ $(x \in A \Rightarrow x \in B)$ <u>and</u> $B \subseteq A$ $(x \in B \Rightarrow x \in A)$.

Proof: ($\subseteq$) Let $x \in A$. Then $x \in \mathbb{Z}$ and $x^2 = 1$. So

$$x^2 - 1 = 0$$
$$(x-1)(x+1) = 0.$$

Thus, $x = 1$ or $x = -1$, so $x \in B$.

($\supseteq$) Let $x \in B$. Then $x = 1$ or $x = -1$. Either way, $x \in \mathbb{Z}$ and $x^2 = 1$, so $x \in A$.

$\blacksquare$

Def: Let A and B be sets. We say A is a **proper subset** of B, written $A \subsetneq B$, if $A \subseteq B$ and $A = B$.

So $A \subsetneq B$ means

$$(\forall x)(x \in A \Rightarrow x \in B) \wedge (\exists y)(y \in B \wedge y \notin A).$$

Warning: Some people use $\subset$ instead of $\subseteq$. $\subset$ does <u>not</u> mean proper subset.