Warm-Up: List all subsets of

- $\{1\}$
- $\{1,2\}$
- $\{1,2,3\}$

Ex: Let $n \in \mathbb{N}$. A set with n elements has exactly 2^n subsets. Why?

Warning: $\in$ vs. $\subseteq$

Ex:
$1 \in \{1,2,3\}$ is true
$\{1\} \in \{1,2,3\}$ is false
$\{1\} \subseteq \{1,2,3\}$ is true
$1 \subseteq \{1,2,3\}$ makes no sense

Ex: $\emptyset \subseteq \emptyset$ (because $\emptyset \subseteq A$ for every set A)
but $\emptyset \notin \emptyset$ (because $x \in \emptyset$ is always false)

Ex: Consider $\{\emptyset\}$, the set whose only element is $\emptyset$.
Then $\emptyset \in \{\emptyset\}$ and $\emptyset \subseteq \{\emptyset\}$.

Thm: ① For all sets A, $A \subseteq A$. [Reflexive]

② For all sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$. [Antisymmetric]

③ For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. [Transitive]

Note: $\leq$ and divisibility have these same 3 properties!

Proof: ① we proved last time.

② is our definition of set equality.

③: Suppose $A \subseteq B$ and $B \subseteq C$. This means

$$x \in A \Rightarrow x \in B \quad \text{is true for every } x$$

and

$$x \in B \Rightarrow x \in C \quad \text{is true for every } x.$$

To prove $A \subseteq C$, suppose $x \in A$ for some x. Then $x \in B$ because $A \subseteq B$. Thus, $x \in C$ because $B \subseteq C$.

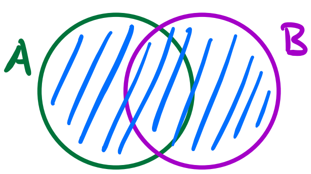
Therefore, $x \in A \Rightarrow x \in C$ for every x, so $A \subseteq C$. $\blacksquare$

Algebra of Sets

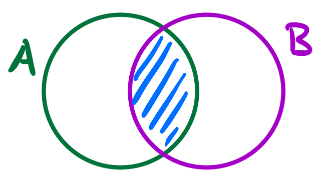
Def: Let A and B be sets.

1. The <u>union</u> of A and B is the set
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$
2. The <u>intersection</u> of A and B is the set
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$
3. The <u>relative complement</u> of B in A is the set
$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$
Also called <u>set difference</u>

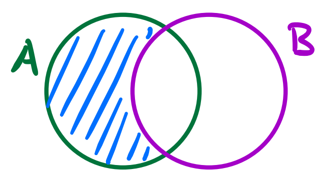
Pictures:



$A \cup B$



$A \cap B$



$A \setminus B$

__Ex__: Let $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$
$= \{2, 4, 6, 8, \dots\}$

and

$P = \{p \in \mathbb{N} \mid p \text{ is prime}\}$
$= \{2, 3, 5, 7, 11, \dots\}$

- $E \cup P = \{n \in \mathbb{N} \mid n \text{ is even or prime}\}$
  $= \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 17, \dots\}$

- $E \cap P = \{n \in \mathbb{N} \mid n \text{ is even and prime}\}$
  $= \{2\}$

- $E \setminus P = \{4, 6, 8, 10, \dots\}$

- $P \setminus E = \{3, 5, 7, 11, \dots\}$

- $\mathbb{N} \setminus E = \{n \in \mathbb{N} \mid n \text{ is odd}\}$
  $= \{1, 3, 5, 7, \dots\}$

- $E \setminus \mathbb{N} = \emptyset$