**Warm-Up**: Let $A$ and $B$ be sets. Show that

$$A \subseteq A \cup B \quad \text{and} \quad B \subseteq A \cup B.$$

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HW 19: You showed $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

**Recall**:
$$x \in A \cup B \iff (x \in A) \vee (x \in B)$$
$$x \in A \cap B \iff (x \in A) \wedge (x \in B)$$

Many theorems from logic translate directly to theorems about sets.

**Lemma**: Let $A$ and $B$ be sets. Then for any object $x$,

(a) $x \notin A \cup B$ if and only if $x \notin A$ and $x \notin B$.

(b) $x \notin A \cap B$ if and only if $x \notin A$ or $x \notin B$.

**Proof**: (a)
$$\begin{aligned} x \notin A \cup B &\iff \neg(x \in A \cup B) \\ &\iff \neg[(x \in A) \vee (x \in B)] \\ &\iff \neg(x \in A) \wedge \neg(x \in B) \quad \text{DeMorgan} \\ &\iff (x \notin A) \wedge (x \notin B). \end{aligned}$$

(b) is similar, using the other DeMorgan Law. $\blacksquare$

**Thm** (DeMorgan Laws for sets):
Let $A, B$, and $S$ be sets. Then

(i) $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$.
(ii) $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$.

**Proof**: (i) We'll show both containments.

($\subseteq$): Let $x \in S \setminus (A \cup B)$. Then $x \in S$ and $x \notin A \cup B$. By the Lemma, $x \notin A$ and $x \notin B$. So $x \in S \setminus A$ and $x \in S \setminus B$. Thus, $x \in (S \setminus A) \cap (S \setminus B)$.

($\supseteq$): Let $x \in (S \setminus A) \cap (S \setminus B)$. Then $x \in S \setminus A$ and $x \in S \setminus B$. So $x \in S$ and $x \notin A$, and $x \in S$ and $x \notin B$. Since $x \notin A$ and $x \notin B$, we have $x \notin A \cup B$ by the Lemma. Thus, because $x \in S$, we have $x \in S \setminus (A \cup B)$.

(ii) is similar. ∎

Similarly, one can prove the following.

<u>Thm</u> (Commutativity of $\cup$ and $\cap$):
Let $A$ and $B$ be sets. Then

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$.

<u>Thm</u> (Associativity of $\cup$ and $\cap$):
Let $A$, $B$, and $C$ be sets. Then

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$.

<u>Thm</u> (Distributive Laws for sets):
Let $A$, $B$, and $S$ be sets. Then

(i) $S \cap (A \cup B) = (S \cap A) \cup (S \cap B)$

(ii) $S \cup (A \cap B) = (S \cup A) \cap (S \cup B)$

# Sets of sets

**Notation:** We'll often use a script letter to denote a set of sets – i.e. a set, all of whose elements are sets.

**Def:** Let $\mathcal{A}$ be a set of sets. Then

- $\bigcup_{A \in \mathcal{A}} A = \{x \mid (\exists A \in \mathcal{A})(x \in A)\}$

- $\bigcap_{A \in \mathcal{A}} A = \{x \mid (\forall A \in \mathcal{A})(x \in A)\}$

**Note:** The book writes $\bigcup \mathcal{A}$ for $\bigcup_{A \in \mathcal{A}} A$

and $\bigcap \mathcal{A}$ for $\bigcap_{A \in \mathcal{A}} A$.

**Ex:** Let $\mathcal{A} = \{\{1,2\}, \{2,3\}, \{2,5,6\}\}$. Then

$$\bigcup_{A \in \mathcal{A}} A = \{1,2\} \cup \{2,3\} \cup \{2,5,6\} = \{1,2,3,5,6\}$$

and

$$\bigcap_{A \in \mathcal{A}} A = \{1,2\} \cap \{2,3\} \cap \{2,5,6\} = \{2\}.$$

Ex: Let $A_n = \{k \in \mathbb{N} \mid k \geq n\}$
$= \{n, n+1, n+2, \dots\}$

So $A_1 = \{1, 2, 3, \dots\} = \mathbb{N}$
$A_2 = \{2, 3, 4, \dots\}$
$A_3 = \{3, 4, 5, \dots\}$
$\vdots$

Set $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$
$= \{A_1, A_2, A_3, \dots\}$.

A set with infinitely many elements, each of which is a set

Then

$$\bigcup_{A \in \mathcal{A}} A = \bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \cdots = \mathbb{N}.$$

Proof: Let $x \in \bigcup_{n=1}^{\infty} A_n$. Then $x \in A_n$ for some $n$. But $A_n \subseteq \mathbb{N}$, so $x \in \mathbb{N}$. Thus, $\bigcup_{i=1}^{\infty} A_i \subseteq \mathbb{N}$.

On the other hand, let $x \in \mathbb{N}$. Since $\mathbb{N} = A_1$, $x \in \bigcup_{n=1}^{\infty} A_n$. Thus, $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$. ∎

Also,

$$\bigcap_{A \in \mathcal{A}} A = \bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots = \emptyset.$$

Proof: Suppose $x \in \bigcap_{n=1}^{\infty} A_n$. Then $x \in A_n$ for every $n$. In particular, $x \in A_1 = \mathbb{N}$. But then $x \notin A_{x+1}$, which contradicts $x \in A_n$ for all $n \in \mathbb{N}$.

So $\bigcap_{n=1}^{\infty} A_n$ must be empty. ∎

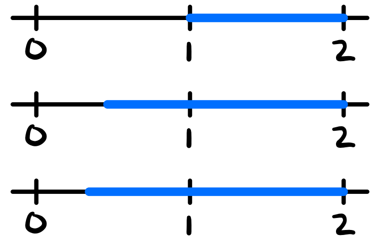
Ex: Let $A_n = [\frac{1}{n}, 2]$ for each $n \in \mathbb{N}$.

$A_1 = [1,2]$

$A_2 = [\frac{1}{2}, 2]$

$A_3 = [\frac{1}{3}, 2]$

$\vdots$




Then $\bigcap_{i=1}^{\infty} A_n = [1,2]$.

Proof: Left to you.

And $\bigcup_{i=1}^{\infty} A_n = (0,2]$.

Proof: Each $A_n \subseteq (0,2]$, so $\bigcup_{n=1}^{\infty} A_n \subseteq (0,2]$

why?

Now, let $x \in (0,2]$.
By the Archimedean Property (Bonus Problem #6), there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < x$.

Thus, $x \in A_m = [\frac{1}{m}, 2]$, and so $x \in \bigcup_{n=1}^{\infty} A_n$.
That is, $(0,2] \subseteq \bigcup_{n=1}^{\infty} A_n$.

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