Warm-Up: Let $\mathcal{A} = \{ \underbrace{[0,n]}_{=\{x\in\mathbb{R} \mid 0\le x\le n\}} \mid n \in \mathbb{N}\}$.

Find $\bigcup_{A\in\mathcal{A}} A$ and $\bigcap_{A\in\mathcal{A}} A$.

Ex: Similarly, if $B_n = (-\frac{1}{n}, 2]$, then

$$\bigcup_{n=1}^{\infty} B_n = (-1,2] \quad \text{and} \quad \bigcap_{n=1}^{\infty} B_n = [0,2].$$

Thm: Let $\mathcal{A}$ be a non-empty set of sets. Let $A_0 \in \mathcal{A}$. Then

$$\bigcap_{A\in\mathcal{A}} A \underset{①}{\subseteq} A_0 \underset{②}{\subseteq} \bigcup_{A\in\mathcal{A}} A.$$

Proof: ① Let $x \in \bigcap_{A\in\mathcal{A}} A$. Then for all $A\in\mathcal{A}$, $x\in A$. In particular, $x\in A_0$. Thus, $\bigcap_{A\in\mathcal{A}} A \subseteq A_0$.

② Let $x\in A_0$. Then there exists some $A\in\mathcal{A}$ such that $x\in A$, because we could take $A = A_0$. This means $x \in \bigcup_{A\in\mathcal{A}} A$. Therefore, $A_0 \subseteq \bigcup_{A\in\mathcal{A}} A$. ∎

Thm (Generalized DeMorgan Laws for sets):

Let S be a set and let $\mathcal{A}$ be a set of sets. Then

(i) $S \setminus \left(\bigcup_{A \in \mathcal{A}} A \right) = \bigcap_{A \in \mathcal{A}} (S \setminus A)$

(ii) $S \setminus \left(\bigcap_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} (S \setminus A)$.

Thm (Generalized Distributive Laws for sets):

Let S be a set and let $\mathcal{A}$ be a set of sets. Then

(i) $S \cap \left(\bigcup_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} (S \cap A)$

(ii) $S \cup \left(\bigcap_{A \in \mathcal{A}} A \right) = \bigcap_{A \in \mathcal{A}} (S \cup A)$.

The Power Set

<u>Def</u>: Let A be a set. The <u>power set</u> of A, denoted $\mathcal{P}(A)$ is the set of all subsets of A.

$$\mathcal{P}(A) = \{S \mid S \subseteq A\}.$$

<u>Ex</u>: $A = \{1,2\}$. Then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.

Cartesian Products

"Def": An ordered pair is a list of two objects in order.
If a and b are objects, then (a,b) denotes the ordered pair with first entry a and second entry b.

What do we mean by "in order"?

Fundamental Property: $(a,b)=(c,d)$ if and only if $a=c$ and $b=d$.

Ex: • If $a \neq b$, then $(a,b) \neq (b,a)$.
• For any a, (a,a) is a perfectly fine ordered pair.

Compare with sets:
• $\{a,b\} = \{b,a\}$
• $\{a,a\} = \{a\}$

Aside: There is an "implementation" of ordered pairs as sets. To do this, define

$$(a,b) = \{\{a\},\{a,b\}\}.$$

Then you can prove that $(a,b)=(c,d) \iff a=c$ and $b=d$.

Cartesian Products

Def: Let A and B be sets. The Cartesian product of A and B is the set

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}.$$

Ex: Let $A = \{a,b,c\}$ and $B = \{2,4\}$. Then

$$A \times B = \{(a,2),\ (a,4),\ (b,2),\ (b,4),\ (c,2),\ (c,4)\}.$$

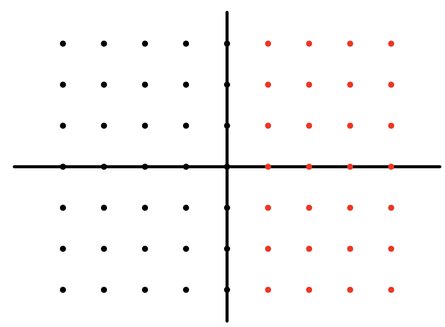
We write $A^2 = A \times A$.

Ex: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$ is the usual Cartesian plane.

Ex: $\mathbb{N} \times \mathbb{Z} = \{(m,n) \mid m \in \mathbb{N},\ n \in \mathbb{Z}\}$.

$\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{(m,n) \mid m \in \mathbb{Z},\ n \in \mathbb{Z}\}$.

Picture:



Note that $\mathbb{N} \times \mathbb{Z} \subseteq \mathbb{Z}^2 \subseteq \mathbb{R}^2$.

For sets A, B, C, we can similarly define

$$A \times B \times C = \{\underbrace{(a,b,c)}_{\text{ordered triples}} \mid a \in A,\ b \in B,\ c \in C\}.$$

More generally, we can define the Cartesian product of n sets to be the set of ordered n-tuples.

Ex: $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x, y, z \in \mathbb{R}\}$.

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n} = \{(x_1, x_2, \dots, x_n) \mid \text{each } x_i \in \mathbb{R}\}.$$