Functions

"Def": Let A and B be sets. A function $f: A \to B$ is a rule which associates to each $x \in A$ an element $f(x) \in B$.

- A is the domain of f, written $A = \mathrm{Dom}(f)$.
  (the set of all valid inputs)
  We might say f is a function on A.

- B is the target or codomain of f.
  (a set containing all possible outputs)

- For $x \in A$, $f(x)$ is the value of f at x.
  [f is the function, $f(x)$ is an element of B]

- The word map is a synonym for function.

Note that to define a function, we must specify both the domain and the target.

Ex: Let $A = \{a, b, c, d\}$. Define $f: A \to \mathbb{Z}$ by

$$f(a) = 2, \quad f(b) = 3, \quad f(c) = 1, \quad f(d) = 1.$$

When the domain is finite, like it is here, we can represent the function as a table.

Dummy variable

x	$f(x)$
a	2
b	3
c	1
d	1

Ex: Consider the functions

Dummy variable

$f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

$g: \mathbb{R} \to [0, \infty)$ " " $g(x) = x^2$

$h: \mathbb{R} \to [-2, \infty)$ " " $h(x) = x^2$

$i: [0, \infty) \to [0, \infty)$ " " $i(x) = x^2$

$j: [1, 2] \to \mathbb{R}$ " " $j(x) = x^2$

$k: [1, 2] \to [1, 4]$ " " $k(x) = x^2$

Q: Why can't we add $\ell: [1, 3] \to [1, 5]$ given by $\ell(x) = x^2$ to this list?

Def: We say two functions f and g are **equal** if

① $\mathrm{Dom}(f) = \mathrm{Dom}(g)$

and

② For every $x \in \mathrm{Dom}(f)$, $\underline{f(x) = g(x)}$.

↑ equality of values

In this case we write $\underline{f = g}$.

↑ equality of functions

Ex: In the previous example, we have 3 functions up to equality: $f = g = h$, i, and $j = k$.

Def: Let $f : A \to B$ be a function. The **range** of f, denoted $\mathrm{Rng}(f)$, is the set

$$\mathrm{Rng}(f) = \{ y \in B \mid f(x) = y \text{ for some } x \in A \}.$$

Note: $\mathrm{Rng}(f) \subseteq B$ automatically.

Informally, $\mathrm{Rng}(f)$ is the set of all function values.