Recall: The <u>range</u> of a function $f: A \to B$ is

$$\mathrm{Rng}(f) = \{ y \in B \mid y = f(x) \text{ for some } x \in A \}.$$

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.
Then $\mathrm{Rng}(f) = [0, \infty)$.

Proof: ($\subseteq$) Let $y \in \mathrm{Rng}(f)$. Then $y \in \mathbb{R}$ and $y = f(x)$ for some $x \in \mathbb{R}$. Thus $y = x^2 \geq 0$, so $y \in [0, \infty)$.

($\supseteq$) Let $y \in [0, \infty)$. Then $y \geq 0$, so $\sqrt{y} \in \mathbb{R}$. Set $x \in \sqrt{y}$. We have

$$f(x) = x^2 = (\sqrt{y})^2 = y,$$

which shows that $y \in [0, \infty)$. $\blacksquare$

Warm-Up: Let $f: \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ be given by

$$f(m, n) = m - n.$$

e.g. $f(1,3) = -2$, $f(3,1) = 2$.

Show that $\mathrm{Rng}(f) = \mathbb{Z}$.

Ex: Let S be any set. Define a function

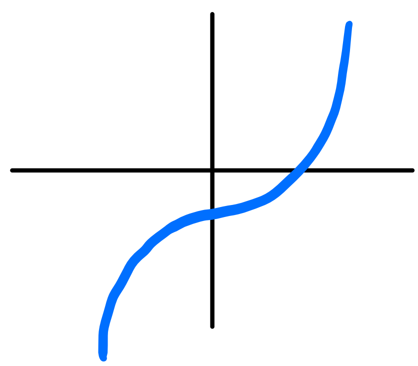
$$\mathrm{id}_S : S \to S$$

by $\mathrm{id}_S(x) = x$ for all $x \in S$.

This is called the <u>identity function</u> on S.

Graphs

Ex: For $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 2$, the graph of f is



What is this? It's

$$\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = x^3 - 2\}.$$

Def: Let $f: A \to B$ be a function.
The *graph* of f is

$$\mathrm{Graph}(f) = \{(x,y) \in A \times B \mid y = f(x)\}.$$

Observe: For each $x \in A$, there is a *unique* $y \in B$ such that $(x,y) \in \mathrm{Graph}(f)$, namely $y = f(x)$.
"vertical line test"

Def: Let A and B be sets. A *function* $f: A \to B$ is a subset

$$\mathrm{Graph}(f) \subseteq A \times B$$

with the property that for all $x \in A$, there exists a unique $y \in B$ such that $(x,y) \in \mathrm{Graph}(f)$.

If $(x,y) \in \mathrm{Graph}(f)$, write $f(x) = y$.

Note:

- You don't have to use this definition. But it's more concrete than defining a function as a "rule".
- We can't always draw Graph(f).
- $\mathrm{Rng}(f) = \{ y \in B \mid (x,y) \in \mathrm{Graph}(f) \}$.

Function Composition

Def: Let $f: A \to B$ and $g: B \to C$ be functions. The <u>composition</u> of g with f is the function

$$g \circ f : A \to C$$

given by

$$(g \circ f)(a) = g(f(a))$$

for all $a \in A$.

"g after f"

Ex: $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2^x$
$g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$.

Then $(g \circ f)(x) = (2^x)^2 = 2^{2x} = 4^x$

and $(f \circ g)(x) = 2^{(x^2)}$

$g \circ f \neq f \circ g$, since $(g \circ f)(1) = 4$
$(f \circ g)(1) = 2$

Order matters!

Note: • Read compositions from right to left

• Sometimes, $g \circ f$ is defined but $f \circ g$ is not

Picture:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Thm: Let $f: A \to B$, $g: B \to C$, and $h: C \to D$ be functions. Then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Proof idea: Both are given by $x \mapsto h(g(f(x)))$.