Surjections

Def: Let $f: A \to B$ be a function.
We say f is a <u>surjection</u> if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Also say: f is <u>surjective</u>, f is <u>onto</u>.

Equivalently: $f: A \to B$ is surjective $\iff$ $\mathrm{Rng}(f) = B$

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$.
Then f is a surjection.

Proof: Let $y \in \mathbb{R}$. Set $x = \sqrt[3]{y} \in \mathbb{R}$. Then

$$f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y.$$

$\blacksquare$

Ex: Let $g: \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2$.
Then g is not surjective.

Proof: Consider $-1 \in \mathbb{R}$. Then for all $x \in \mathbb{R}$,
$g(x) = x^2 \neq -1$.

Note: f surjective $\Leftrightarrow (\forall y \in B)(\exists x \in A)(f(x) = y)$

f not surjective $\Leftrightarrow (\exists y \in B)(\forall x \in A)(f(x) \neq y)$.

Ex: However, $h : \mathbb{R} \to [0, \infty)$ given by $h(x) = x^2$ is surjective.

Injections

Def: A function $f : A \to B$ is an *injection* if for all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Also say: f is *injective*, f is *one-to-one*.

Ex: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is an injection.

Proof: Let $x_1, x_2 \in \mathbb{R}$ and suppose $x_1^3 = x_2^3$. Then $\sqrt[3]{x_1^3} = \sqrt[3]{x_2^3}$, i.e. $x_1 = x_2$. ∎

Ex: $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ is not an injection.

Proof: Consider $-1, 1 \in \mathbb{R}$. Then $-1 \neq 1$, but
$g(-1) = (-1)^2 = 1 = (1)^2 = g(1)$. ∎

Note: f injective $\iff (\forall x_1, x_2 \in A)[\underset{T}{f(x_1)=f(x_2)} \Rightarrow \underset{F}{x_1 = x_2}]$

$\iff (\forall x_1, x_2 \in A)[\underset{T}{x_1 \neq x_2} \Rightarrow \underset{F}{f(x_1) \neq f(x_2)}]$

f not injective $\iff (\exists x_1, x_2 \in A)[x_1 \neq x_2 \text{ and } f(x_1) = f(x_2)]$

Ex: However, $h: [0, \infty) \to \mathbb{R}$ given by $h(x) = x^2$ is injective.

Bijections

Def: A function $f: A \to B$ is a bijection if it is both a surjection and an injection.

f surjective $\iff (\forall y \in B)(\exists x \in A)[f(x) = y]$

f injective $\iff (\forall x_1, x_2 \in A)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$

$\iff (\forall y \in B)(\forall x_1, x_2 \in A)[(f(x_1) = y \wedge f(x_2) = y) \Rightarrow x_1 = x_2]$

Together, we get the following:

Lemma: Let $f: A \to B$ be a function. Then f is a bijection if and only if for every $y \in B$, there exists a unique $x \in A$ such that $f(x) = y$.

Ex: $f: \mathbb{Z} \to \mathbb{Z}$, $n \longmapsto n+1$. For each $y \in \mathbb{Z}$, there is a unique $x \in \mathbb{Z}$ such that $f(x) = y$, namely $x = y - 1$.

Inverse Functions

A bijection $f: A \to B$ gives us a rule for going back to B from A. Specifically, $y \in B$ can map back to the unique $x \in A$ such that $f(x) = y$.

Def: Let $f: A \to B$ be a bijection. The <u>inverse function</u> of f is

$$f^{-1}: B \to A$$

defined as follows: For each $y \in B$, $f^{-1}(y)$ is the unique element $x \in A$ such that $f(x) = y$.

That is $f^{-1}(y) = x \iff y = f(x)$.

Ex: $f: \mathbb{R} \to (0, \infty)$ given by $f(x) = e^x$ is a bijection.

$f^{-1}: (0, \infty) \to \mathbb{R}$ is given by $f^{-1}(y) = \ln(y)$.

$$\ln(y) = x \iff y = e^x$$

Ex: $g : [0,\infty) \longrightarrow [0,\infty)$ is a bijection.
$x \longmapsto x^2$

Its inverse is $g^{-1} : [0,\infty) \to [0,\infty)$
$y \longmapsto \sqrt{y}$

$\sqrt{y} = x \iff y = x^2$ and $x \geq 0$

Ex: $\sin : \mathbb{R} \to \mathbb{R}$ is not a bijection,
but $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [1,1]$ is.

Its inverse is $\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin^{-1}(y) = x \iff y = \sin(x)$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$