Warm-Up: Prove that

$$f : \mathbb{R} \setminus \{3\} \to \mathbb{R} \setminus \{1\}$$
$$x \mapsto \frac{x}{x-3}$$

is a bijection.

Recall: A bijection $f: A \to B$ has an inverse function $f^{-1}: B \to A$, such that

$$f^{-1}(y) = x \iff f(x) = y$$

Ex: $\sin: \mathbb{R} \to \mathbb{R}$ is not a bijection, but $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [1,1]$ is.

Its inverse is $\sin^{-1}: [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\sin^{-1}(y) = x \iff y = \sin(x) \text{ and } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Thm: Let $f: A \to B$ be a bijection and let $f^{-1}: B \to A$ be its inverse. Then

$$① \; f^{-1} \circ f = id_A : A \to A$$

and

$$② \; f \circ f^{-1} = id_B : B \to B$$

This is essentially a rephrasing of the fundamental identity $f^{-1}(y) = x \iff f(x) = y$.

Proof: ① Let $x \in A$. We must show

$$(f^{-1} \circ f)(x) = id_A(x) = x.$$

Set $y = f(x)$. Then, by definition of f^{-1}, $f^{-1}(y) = x$. But then

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x. \quad \checkmark$$

② Let $y \in B$ We must show

$$(f \circ f^{-1})(y) = id_B(y) = y.$$

Set $x = f^{-1}(y)$. Then $f(x) = y$, so

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y. \quad \checkmark$$

$\blacksquare$

Cor: Let $f: A \to B$ be a bijection. Then its inverse $f^{-1}: B \to A$ is also a bijection, and $(f^{-1})^{-1} = f$.

Proof: Let $f: A \to B$ be a bijection.

- **f^{-1} is surjective**: Let $x \in A$.

  We must find $y \in B$ so that $f^{-1}(y) = x$.

  Set $y = f(x)$. Then, by the theorem,

  $$f^{-1}(y) = f^{-1}(f(x)) = x.$$

  ✓

- **f^{-1} is injective**: Let $y_1, y_2 \in B$ such that $f^{-1}(y_1) = f^{-1}(y_2)$.

  Then

  $$f(f^{-1}(y_1)) = f(f^{-1}(y_2)),$$

  so by the theorem,

  $$y_1 = y_2.$$

  ✓

- **$(f^{-1})^{-1} = f$**: By definition, for $x \in A$ and $y \in B$,

  $$(f^{-1})^{-1}(x) = y \iff x = f^{-1}(y) \iff f(x) = y.$$

  Thus, $(f^{-1})^{-1} = f$.

∎

The following theorems are proved using similar methods.

Thm: Let $f: A \to B$ and $g: B \to A$ be functions. If
$$g \circ f = id_A \quad \text{and} \quad f \circ g = id_B,$$
then f is a bijection and $g = f^{-1}$.

Thm: If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection also, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Cardinality

What does it mean for a set A to have exactly n elements?

Ex: $A = \{4, \text{red}, \$\}$ has exactly 3 elements

How do we know? We can list them:

1. 4
2. red
3. \$

This is just a bijection $f : \{1, 2, 3\} \to A$.

surjection $\Leftrightarrow$ every element in A is on the list

injection $\Leftrightarrow$ no element in A is on the list more than once.

Def: Let A and B be sets. We say A and B <u>have the same cardinality</u>, denoted $|A| = |B|$, if there exists a bijection $f : A \to B$.

Book: A and B are <u>equinumerous</u>, $\bar{\bar{A}} = \bar{\bar{B}}$.

This is an equivalence relation.

Thm: Let A, B, C be sets. Then

① $|A| = |A|$. [Reflexive]

② If $|A| = |B|$, then $|B| = |A|$. [Symmetric]

③ If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. [Transitive]

Proof sketch: ① $\mathrm{id}_A : A \to A$, $x \mapsto x$ is a bijection.

② If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.

③ If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection. ∎

If A is a set and $n \in \mathbb{N}$ such that A and $\{1, 2, \ldots, n\}$ have the same cardinality, then we say A has <u>cardinality n</u> (or A has <u>exactly n elements</u>), and write $|A| = n$.

We also write $|\emptyset| = 0$.