Warm-Up: Prove that the function

$$f: \mathbb{N} \longrightarrow \mathbb{N}\setminus\{1\}$$
$$x \longmapsto x+1$$

is a bijection.

Finite and infinite sets

Def: A set A is **finite** if either

- $A = \emptyset$ (i.e. $|A| = 0$)

or

- there exists $n \in \mathbb{N}$ such that $|A| = n$.

A set is **infinite** if it is not finite.

Ex: $A = \{4, \text{red}, \$\}$. $|A| = 3$

Ex: $B = \{a, b, c, \dots, z\}$. $|B| = 26$

Ex: Is there a set C such that $|C| = 3$ and $|C| = 26$? No!

<u>Thm</u>: Let $n, m \in \mathbb{N} \cup \{0\}$. If A is a set such that $|A| = n$ and $|A| = m$, then $n = m$.

<u>Proof idea</u>:

$|A| = n$ means there is a bijection $f: \{1, \dots, n\} \to A$
$|A| = m$ " " " $g: \{1, \dots, m\} \to A$.

So $g^{-1} \circ f : \{1, \dots, n\} \to \{1, \dots, m\}$ is a bijection.

- If $n > m$, this can't be injective.
- If $n < m$, this can't be surjective.

<u>Ex</u>: $\mathbb{N}$ is infinite

Why? Let $n \in \mathbb{N}$ and suppose $f: \{1, \dots, n\} \to \mathbb{N}$ is a function. Set

$$m = \text{maximum of } f(1), f(2), \dots, f(n).$$

Then for all $i \in \{1, \dots, n\}$, $f(i) \le m < m+1$.
So $f(i) \neq m+1$, showing that f is not surjective.
So f cannot be a bijection.

Ex: Similarly, $\mathbb{Q}$ and $\mathbb{R}$ are infinite.

WARNING: It may be tempting to write

$$|\mathbb{N}| = \infty$$
$$|\mathbb{Q}| = \infty$$
$$|\mathbb{R}| = \infty$$

We <u>will not</u> do this.

As we will see, $|\mathbb{N}| = |\mathbb{Q}|$, but $|\mathbb{N}| \neq |\mathbb{R}|$.

First, more on finite sets.

Thm: Let S be a <u>finite</u> set and $T \subseteq S$. Then

- T is finite
- $|T| \leq |S|$
- $|T| = |S|$ if and only if $T = S$.

Proof: Book Thms 13.30, 13.33.

Cor: Let A and B be finite sets, and let $f: A \to B$ be a function. Then

① If f is an injection, then $|A| \leq |B|$

② If f is a surjection, then $|A| \geq |B|$

Proof: ① Suppose $f: A \to B$ is injective. Then

$$f: A \to \mathrm{Rng}(f)$$

is a bijection. Hence, $|A| = |\mathrm{Rng}(f)|$.
But $\mathrm{Rng}(f) \subseteq B$, so $|\mathrm{Rng}(f)| \leq |B|$ by the Theorem. Together, we get $|A| \leq |B|$.

② Suppose $f: A \to B$ is surjective. Since B is finite, $|B| = n$ for some $n \in \mathbb{N}$, so we can write

$$B = \{b_1, b_2, \ldots, b_n\}.$$

For each $i \in \{1, \ldots, n\}$, let $a_i \in A$ be such that $f(a_i) = b_i$.

If $i \neq j$, then $f(a_i) = b_i \neq b_j = f(a_j)$, so $a_i \neq a_j$.

Thus, $|\{a_1, \dots, a_n\}| = n$. But $\{a_1, \dots, a_n\} \subseteq A$, so $n \leq |A|$. Since $|B| = n$, we have $|A| \geq |B|$.

$\blacksquare$

The contrapositive of ① is the

<u>Pigeonhole Principle</u>: Let A and B be finite sets and $f: A \to B$ a function. If $|A| > |B|$, then f is not injective.

A - set of pigeons

B - set of pigeonholes

$f: A \to B$ puts each pigeon in a pigeonhole

Then there is a pigeonhole containing more than one pigeon.

<u>Ex</u>: If $a_1, a_2, a_3, a_4 \in \mathbb{Z}$, then the difference $a_i - a_j$ will be divisible by 3 for some $i \neq j$.

Ex: Suppose n people are at a party. Then there are two people who have the same number of friends at the party.