# Infinite Sets

We already saw that

$$f: \mathbb{N} \longrightarrow \mathbb{N} \setminus \{1\}$$
$$x \longmapsto x+1$$

is a bijection, so $|\mathbb{N}| = |\mathbb{N} \setminus \{1\}|$.

Here's another example:

**Ex:** Let $E = \{n \in \mathbb{N} \mid n \text{ is even}\} = \{2, 4, 6, 8, \ldots\}$.
Then

$$g: \mathbb{N} \rightarrow E$$
$$x \mapsto 2x$$

is a bijection. Thus, $|\mathbb{N}| = |E|$.

**Proof:** Let $x_1, x_2 \in \mathbb{N}$. If $f(x_1) = f(x_2)$, then $2x_1 = 2x_2$, so cancelling the 2 gives $x_1 = x_2$. Thus, $f$ is injective.

Let $y \in E$. Then $y = 2k$ for some $k \in \mathbb{N}$ (why?) Thus, $f(k) = 2k = y$. This shows that $f$ is surjective. ∎

**Def:** A set $A$ is <u>countably infinite</u> if there exists a bijection $f: \mathbb{N} \to A$. That is, if $|A| = |\mathbb{N}|$.

A set is <u>countable</u> if it is finite or countably infinite.

A set is <u>uncountable</u> if it is not countable.

**Ex:** $\mathbb{N}$ is countably infinite

$\mathbb{N} \setminus \{1\}$ is countably infinite

$E = \{n \in \mathbb{N} \mid n \text{ is even}\}$ is countably infinite.

$\mathbb{Z}$ is countably infinite

**Think:** Countably infinite sets can be enumerated in an infinite list.

**Thm:** Let $A$ be a countably infinite set. Then any subset $B \subseteq A$ is countable.

## Ex: $\mathbb{N} \times \mathbb{N}$ is countably infinite

**Key:** Write the elements of $\mathbb{N} \times \mathbb{N}$ in a grid

| | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|
| 1 | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(1,5)$ | |
| 2 | $(2,1)$ | $(2,2)$ | $(2,3)$ | $(2,4)$ | $(2,5)$ | |
| 3 | $(3,1)$ | $(3,2)$ | $(3,3)$ | $(3,4)$ | $(3,5)$ | |
| 4 | $(4,1)$ | $(4,2)$ | $(4,3)$ | $(4,4)$ | $(4,5)$ | |
| 5 | $(5,1)$ | $(5,2)$ | $(5,3)$ | $(5,4)$ | $(5,5)$ | |
| $\vdots$ | | | | | | $\ddots$ |

Define a bijection $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by reading along the northeast diagonals in order:

$$f(1) = (1,1)$$
$$f(2) = (2,1)$$
$$f(3) = (1,2)$$
$$f(4) = (3,1)$$
$$\vdots$$

**Ex:** The set $\mathbb{Q}_{>0} = \{ q \in \mathbb{Q} \mid q > 0 \}$ of positive rational numbers is countably infinite.

**Key idea:** Each $q \in \mathbb{Q}_{>0}$ can be written uniquely as $q = \frac{a}{b}$ where

- $a, b \in \mathbb{N}$

and

- $\frac{a}{b}$ is in lowest terms $(\gcd(a,b) = 1)$

Now, use a grid again, but cross out fractions not in lowest terms:

| | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|
| 1 | $\frac{1}{1}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | |
| 2 | $\frac{2}{1}$ | ~~$\frac{2}{2}$~~ | $\frac{2}{3}$ | ~~$\frac{2}{4}$~~ | $\frac{2}{5}$ | |
| 3 | $\frac{3}{1}$ | $\frac{3}{2}$ | ~~$\frac{3}{3}$~~ | $\frac{3}{4}$ | $\frac{3}{5}$ | |
| 4 | $\frac{4}{1}$ | ~~$\frac{4}{2}$~~ | $\frac{4}{3}$ | ~~$\frac{4}{4}$~~ | $\frac{4}{5}$ | |
| 5 | $\frac{5}{1}$ | $\frac{5}{2}$ | $\frac{5}{3}$ | $\frac{5}{4}$ | ~~$\frac{5}{5}$~~ | |
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Define a bijection $g: \mathbb{N} \to \mathbb{Q}_{>0}$ by reading the remaining entries along the northeast diagonals.

$g(1) = 1, \; g(2) = 2, \; g(3) = \frac{1}{2}, \; g(4) = 3, \; g(5) = \frac{1}{3}, \; \ldots$

Ex: $\mathbb{Q}$ is countably infinite.

Let $g: \mathbb{N} \to \mathbb{Q}_{\geq 0}$ be the bijection above. Define a bijection $h: \mathbb{N} \to \mathbb{Q}$ by

$$h(n) = \begin{cases} 0 & \text{if } n = 1 \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ -g\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

So

$$h(1) = 0$$
$$h(2) = g(1) = 1$$
$$h(3) = -g(1) = -1$$
$$h(4) = g(2) = 2$$
$$h(5) = -g(2) = -2$$
$$\vdots$$

**Thm:** $|\mathbb{R}| \neq |\mathbb{N}|$ ($\mathbb{R}$ is uncountable)

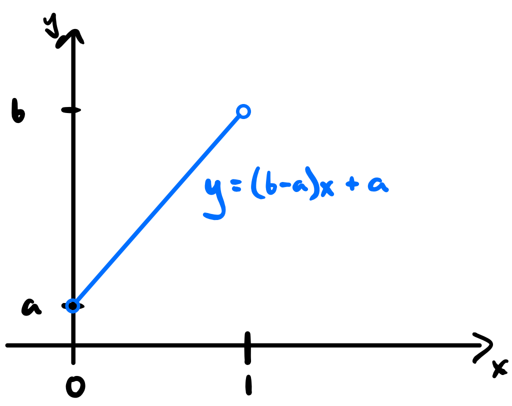
**Step 1:** If $a, b \in \mathbb{R}$ with $a < b$, then $|(a,b)| = |(0,1)|$.

We must give a bijection between $(0,1)$ and $(a,b)$.

A linear function will work:

$$f : (0,1) \to (a,b)$$
$$x \mapsto (b-a)x + a$$

**Graph:**



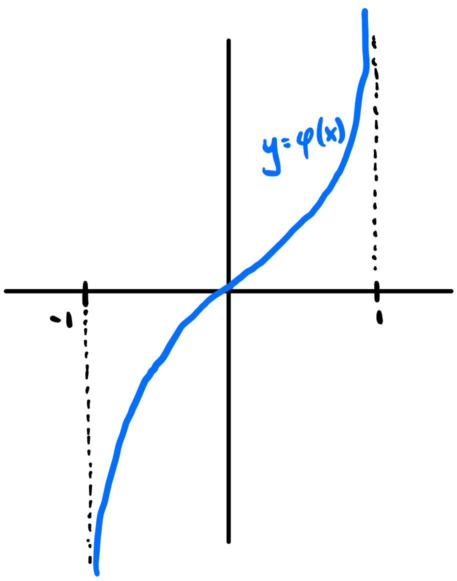

**Exercise:** Check that $f$ is a bijection.

## Step 2: $|\mathbb{R}| = |(-1,1)|$

There are many ways to do this, but we'll use the book's: Define

$$\varphi : (-1,1) \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{x}{1-|x|}$$

Graph:




Exercise: Check that $\varphi$ is a bijection.
(Follows from HW 23 practice problems)

**Step 3:** There is no surjection $\mathbb{N} \to (0,1)$ (and thus no bijection $\mathbb{N} \to (0,1)$).

Why is this enough? If $|\mathbb{N}| = |\mathbb{R}|$, then since $|\mathbb{R}| = |(-1,1)|$ and $|(-1,1)| = |(0,1)|$, transitivity gives $|\mathbb{N}| = |(0,1)|$, a contradiction.

To show this, we use Cantor's Diagonal Argument.

Need: • Every real number has an infinite decimal representation.

e.g.

$$\frac{1}{3} = 0.3333333\cdots$$
$$\frac{3}{4} = 0.7500000\cdots$$
$$\pi - 3 = 0.14159265\cdots$$

• This representation is unique if we don't allow infinite repeating 9s.

e.g.

$$\frac{3}{4} = 0.749999999\cdots = 0.750000000\cdots$$

Now, let $f: \mathbb{N} \to (0,1)$ be a function.
Think of this as an infinite list:

$$c_1 = f(1) = 0.x_{11}x_{12}x_{13}x_{14}x_{15}\cdots$$
$$c_2 = f(2) = 0.x_{21}x_{22}x_{23}x_{24}x_{25}\cdots$$
$$c_3 = f(3) = 0.x_{31}x_{32}x_{33}x_{34}x_{35}\cdots$$
$$c_4 = f(4) = 0.x_{41}x_{42}x_{43}x_{44}x_{45}\cdots$$
$$\vdots$$

$x_{nm}$ = mth digit of nth number

Define a number $c_0$ by

$$c_0 = 0.x_{01}x_{02}x_{03}x_{04}x_{05}\cdots$$

where

$$x_{0m} = \begin{cases} 1 & \text{if } x_{mm} \neq 1 \\ 2 & \text{if } x_{mm} = 1 \end{cases}$$

Then $c_0 \in (0,1)$, but

| | | |
|---|---|---|
| $c_0 \neq c_1$ | because | $x_{01} \neq x_{11}$ |
| $c_0 \neq c_2$ | " | $x_{02} \neq x_{22}$ |
| $c_0 \neq c_3$ | " | $x_{03} \neq x_{33}$ |
| $\vdots$ | | |

Thus, $c_0 \notin Rng(f)$, so $f$ is not surjective.