Warm-Up: Use a truth table to show that $P \Rightarrow Q$ is <u>not</u> logically equivalent to $Q \Rightarrow P$.

Recall: $P \Rightarrow Q \equiv \neg P \vee Q$.

Corollary: $\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$

Proof:

$$\neg(P \Rightarrow Q) \equiv \neg(\neg P \vee Q)$$

$$\equiv \neg(\neg P) \wedge \neg Q \quad \text{(DeMorgan)}$$

$$\equiv P \wedge \neg Q \quad \text{(Double negation)}$$

Ex: The negation of

"If it is Wednesday, then I will attend class"

is logically equivalent to

"It is Wednesday and I will not attend class."

A sentence of the form $P \Rightarrow Q$ is called a <u>conditional sentence</u>.

Ways to say $P \Rightarrow Q$:

"P implies Q"
"If P, then Q"
"P is sufficient for Q"
"Q is necessary for P"

In a conditional sentence $P \Rightarrow Q$, P is the <u>antecedent</u> and Q is the <u>consequent</u>.

More informally, P is the "assumption" and Q is the "conclusion."

Converse and contrapositive

Let P and Q be sentences.

The <u>converse</u> of $P \Rightarrow Q$ is the sentence

$$Q \Rightarrow P.$$

The <u>contrapositive</u> of $P \Rightarrow Q$ is the sentence

$$\neg Q \Rightarrow \neg P.$$

<u>Ex</u>: "If it is raining, then the ground is wet."

Converse: "If the ground is wet, then it is raining."

Contrapositive: "If the ground is dry, then it is not raining."

We saw in the Warm-Up that $P \Rightarrow Q$ is <u>not</u> logically equivalent to the converse $Q \Rightarrow P$.

<u>Thm</u>: $P \Rightarrow Q$ is logically equivalent to the contrapositive $\neg Q \Rightarrow \neg P$.

<u>Proof</u>: We have

$$\begin{aligned}\neg Q \Rightarrow \neg P &\equiv \neg(\neg Q) \vee \neg P \\ &\equiv Q \vee \neg P \\ &\equiv \neg P \vee Q \\ &\equiv P \Rightarrow Q.\end{aligned}$$

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Alternatively:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

A final logical connective:

(5) Biconditional: $\Leftrightarrow$ means "if and only if"

$P \Leftrightarrow Q$ is true exactly when P and Q have the same truth value.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Thm: $P \Leftrightarrow Q$ is logically equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

Proof:

P	Q	$P \Leftrightarrow Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

∎

A sentence of the form $P \Leftrightarrow Q$ is called a biconditional sentence.

Ways to say $P \Leftrightarrow Q$:

"P if and only if Q"

"P is necessary and sufficient for Q"

"Q is necessary and sufficient for P"

"P is necessary for Q" is $Q \Rightarrow P$

"P is sufficient for Q" is $P \Rightarrow Q$