Warm-Up: Let x and y be real numbers.

Let S be the conditional sentence

"If $xy > 0$, then $x > 0$ and $y > 0$."

- Is S true or false?
- Write the converse and contrapositive of S. Are these sentences true or false?
- Write the negation $\neg S$ as an "and" sentence.

Recall: A biconditional sentence $P \Leftrightarrow Q$ is true precisely when P and Q have the same truth values.

$$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Ex: $x^2 = 9 \iff x = 3$ or $x = -3$

This sentence is true. Why?

Let P be "$x^2 = 9$" and Q be "$x = 3$ or $x = -3$".

We'll show $P \Rightarrow Q$ and $Q \Rightarrow P$ are both true.

<u>$P \Rightarrow Q$</u>

P	Q	P ⇒ Q
T	T	T
T	F	F
F	T	T
F	F	T

<u>Case 1</u>: P is true. Then $x^2 = 9$, so

$$x^2 - 9 = 0.$$

Factor to get $(x-3)(x+3) = 0$.

Hence, $x - 3 = 0$ or $x + 3 = 0$.
That is, $x = 3$ or $x = -3$, so Q is true. ✓

<u>Case 2</u>: P is false. Then $P \Rightarrow Q$ is vacuously true. ✓

$\underline{Q \Rightarrow P}$

<u>Case 1</u>: Q is true. Then $x=3$ or $x=-3$, so

$$x^2 = 3^2 = 9 \quad \text{or} \quad x^2 = (-3)^2 = 9.$$

That is, P is true. ✓

<u>Case 2</u>: Q is false. Then $Q \Rightarrow P$ is vacuously true. ✓

∎

Conditional Proof

In general, to show $P \Rightarrow Q$ is true, we must

1. Assume P is true.
2. Under this assumption, show that Q must be true also.

Why is this valid?

When P is false, $P \Rightarrow Q$ is automatically true.

This method is called <u>conditional proof</u>.

Most of our theorems will be of the form $P \Rightarrow Q$, so we will write a lot of conditional proofs.

To prove a biconditional $P \Leftrightarrow Q$, we need two conditional proofs: for $P \Rightarrow Q$ and $Q \Rightarrow P$.

Tautologies

A sentence is called a <u>tautology</u> if it is always true for structural reasons (i.e. because of how it is constructed using $\neg$, $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$).

<u>Ex</u>: "Modus ponens"

$$(P \Rightarrow Q) \wedge P \Rightarrow Q$$

is a tautology.

<u>Proof</u>: Assume $(P \Rightarrow Q) \wedge P$ is true.

[Again, if it's false there's nothing to do.]

Then both of $P \Rightarrow Q$ and P are true, so Q must be true. ∎

Modus ponens is a common step in logical reasoning.

Ex: If it is raining, then the ground is wet. It is raining.

Therefore, the ground is wet.