Warm-Up: Show that the sentence

$$(P \wedge Q) \Rightarrow P$$

is a tautology — that is, it is true regardless of the sentences P and Q.

Do this ① Using a truth table.

② By writing a conditional proof.

Another perspective on conditional proof:

To prove $A \Rightarrow C$ is true, it is enough to show that it is impossible for A to be true and C to be simultaneously false.

To do this, either

- Assume A is true and show C must also be true.

OR

- Assume C is false and show A must also be false. [i.e. prove $\neg C \Rightarrow \neg A$]

More tautologies

Ex: $P \vee \neg P$ is a tautology.

"The law of the excluded middle."

Ex: Show that the sentence

$$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$$

is a tautology. "Hypothetical syllogism"

Proof: Suppose $(P \Rightarrow Q) \wedge (Q \Rightarrow R)$ is true. Thus, $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true.

We want to argue that $P \Rightarrow R$ must be true.

Assume P is true. [If not, $P \Rightarrow Q$ is true vacuously and we are done.]

Since $P \Rightarrow Q$ and P are both true, Q must be true.

Since $Q \Rightarrow R$ and Q are both true, R must be true.

Therefore, $P \Rightarrow R$ is true, as desired. $\blacksquare$

<u>Recall</u>: The truth value of the sentence "If $xy > 0$, then $x > 0$ and $y > 0$" depends on the numerical values of x and y.

We said it is false because it <u>can</u> be false (e.g. $x = -1$, $y = -1$), but it can also be true (e.g. $x = 1$, $y = 1$ or $x = -1$, $y = 1$).

Quantifiers let us discuss such situations.

Quantifiers

The **universal quantifier** is $\forall$, which means "for all."

If $P(x)$ is a sentence involving the variable x, then $(\forall x)\,P(x)$ is the sentence

"for all x, $P(x)$"

also read as

"for every x, $P(x)$"
"for each x, $P(x)$"
"for any x, $P(x)$."

The <u>existential quantifier</u> is $\exists$, which means "there exists."

$(\exists x)\, P(x)$ is the sentence

"there exists x such that $P(x)$"

also read as

"for some x, $P(x)$"

"for at least one x, $P(x)$."

<u>Note</u>:
- A quantifier $(\forall, \exists)$ is always "attached" to a variable, called the <u>bound variable</u>.
- A quantifier is always followed by a sentence involving the bound variable.

Ex: Let $P(x)$ be the sentence

$$\text{"}(x > 1) \Rightarrow (x^2 > 1)\text{"}$$

and let $Q(x)$ be the converse

$$\text{"}(x^2 > 1) \Rightarrow (x > 1)\text{."}$$

Then
- $(\forall x)\, P(x)$ is true.
- $(\exists x)\, P(x)$ is true.
- $(\forall x)\, Q(x)$ is false.
- $(\exists x)\, Q(x)$ is true.

Note: We should be more careful to specify which values the bound variable is allowed to take on.

The above statements are correct when x can be any real number.

To indicate this, we will write $(\forall x \in \mathbb{R})$ and $(\exists x \in \mathbb{R})$.