Warm-Up: For each sentence, draw a number line and indicate all x-values making the sentence true.

(a) $(x>2) \wedge (x^2>4)$

(b) $(x>2) \vee (x^2>4)$

(c) $(x>2) \Rightarrow (x^2>4)$

(d) $(x>2) \Leftrightarrow (x^2>4)$

Last time: $\forall$ and $\exists$.

Ex: Which statements are true?

① $(\exists x \in \mathbb{R})(x+4=9)$

True: $x=5$.

② $(\exists x \in \mathbb{R})[(x+4=9) \wedge (x \neq 5)]$

False: $x+4=9 \Rightarrow x=9-4=5$

③ $(\forall x \in \mathbb{R})(x+4=9)$

False: Try $x=0$.

④ $(\exists\, x \in \mathbb{R})(x^2 + 6x + 8 \geq 0)$

True: Try $x = 0$.

⑤ $(\forall\, x \in \mathbb{R})(x^2 + 6x + 8 \geq 0)$

Can guess and check, or complete the square:

$$x^2 + 6x + 8 = x^2 + 6x + 9 - 1$$
$$= (x+3)^2 - 1.$$

False: Try $x = -3$.

⑥ $(\forall x \in \mathbb{R})(x^2 + 6x + 10 \geq 0)$

True: $x^2 + 6x + 10 = (x+3)^2 + 1 \geq 1 > 0$ for all real numbers x.

Observe:
- A single example proves a $\exists$ statement.
- A single counterexample disproves a $\forall$ statement.
- To prove a $\forall$ statement or disprove a $\exists$ statement, we need an argument that works for all values.

Free + Bound Variables

Let $P(x) =$ "$x^2 + 6x + 8 \geq 0.$"

- Is $P(x)$ true? It depends on x.

  We say that x is a <u>free variable</u> in the sentence $P(x)$.

  <u>Think</u>: The sentence $P(x)$ is a function of x.

- Is $(\forall x \in \mathbb{R})\, P(x)$ true? No!

  This sentence does NOT depend on x, because of the quantifier $\forall$.

  In this case, we say x is a <u>bound variable</u> in the sentence $(\forall x \in \mathbb{R})\, P(x)$.

The quantifier $\exists$ can also bound variables: $(\exists x)\, P(x)$ does not depend on x.

<u>Analogy</u>: $f(x) = x^2$ vs. $\int_0^1 x^2 \, dx$

Note: When we use a quantifier ($\forall$ or $\exists$), a bound variable is ranging over a **universe** of possibilities.

Usually, we should be explicit about this.

Common choices:

$\mathbb{Z}$ = the set of integers
$\mathbb{Q}$ = the set of rational numbers
$\mathbb{R}$ = the set of real numbers
$\mathbb{C}$ = the set of complex numbers

The universe matters!

Ex:

$(\exists x)(x^2 = 2)$ Ambiguous

$(\exists x \in \mathbb{Z})(x^2 = 2)$ False, $\sqrt{2} \notin \mathbb{Z}$

$(\exists x \in \mathbb{R})(x^2 = 2)$ True, $\sqrt{2} \in \mathbb{R}$

Ex:

$(\forall x \in \mathbb{R})(x^2 \geq 0)$ True

$(\forall x \in \mathbb{C})(x^2 \geq 0)$ False, $\sqrt{-1} \in \mathbb{C}$

Note: Over a finite set (universe),

- $\forall$ is an "and" statement
- $\exists$ is an "or" statement

Ex: If $A = \{-3, 1, 4\}$, then

$$(\forall x \in A)(x^2 < 20) \equiv ((-3)^2 < 20) \wedge (1^2 < 20) \wedge (4^2 < 20)$$

$$(\exists x \in A)(x > 0) \equiv (-3 > 0) \vee (1 > 0) \vee (4 > 0)$$

(Both true)

For this reason, we can think of $\forall$ as "generalized and" and $\exists$ as "generalized or."

Thm (Generalized DeMorgan's Laws)

(a) $\neg[(\forall x \in A)\ P(x)] \equiv (\exists x \in A)(\neg P(x))$

(b) $\neg[(\exists x \in A)\ P(x)] \equiv (\forall x \in A)(\neg P(x))$

Proof: (a) Suppose $\neg[(\forall x \in A)\ P(x)]$ is true.

Then $(\forall x \in A)\ P(x)$ is false.

So there is some $x_0 \in A$ such that $P(x_0)$ is false, i.e. $\neg P(x_0)$ is true.

Hence $(\exists x \in A)(\neg P(x))$ is true.

Conversely, suppose $(\exists x \in A)(\neg P(x))$ is true.

Then there is $x_0 \in A$ such that $\neg P(x_0)$ is true, i.e. $P(x_0)$ is false.

So $(\forall x \in A)\ P(x)$ is false. Therefore, $\neg(\forall x \in A)\ P(x)$ is true.

(b) is similar (see book). $\blacksquare$