Warm-Up: What is the difference between

(a) $(\exists x \in \mathbb{R})(\cos x = 0 \text{ and } \tan x = 0)$

and

(b) $(\exists x \in \mathbb{R})(\cos x = 0)$ and $(\exists x \in \mathbb{R})(\tan x = 0)$?

Ex: Let $P(x)$ be a sentence depending on $x \in A$. Then

$$(\forall x \in A)\, P(x) \implies (\exists x \in A)\, P(x)$$

is a tautology.

Thm (Generalized Distributive Laws):

Let P be a sentence not involving x.

Let $Q(x)$ be a sentence involving x.

Then

a) $P \wedge [(\exists x \in A)\, Q(x)] \equiv (\exists x \in A)[P \wedge Q(x)]$

b) $P \vee [(\forall x \in A)\, Q(x)] \equiv (\forall x \in A)[P \vee Q(x)]$.

Proof: Omitted (see book).

Order of Quantifiers

Suppose $P(x,y)$ is a sentence involving 2 variables. What is the difference between

(a) $(\forall x)[(\exists y)\, P(x,y)]$

and

(b) $(\exists y)[(\forall x)\, P(x,y)]$?

implicit

Ex: $P(x,y) =$ " $x+y=1$ "

(a) is " for any $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x+y=1$ " True!

Proof: Let $x \in \mathbb{R}$. Set $y = 1-x$. Then $y \in \mathbb{R}$ and $x+y = x+(1-x) = 1$. ∎

(b) is " there is $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$, we have $x+y=1$ " False!

How to prove? Let's show $\neg$(b) is true.

By DeMorgan,

$$\neg(\exists y)[(\forall x)\,P(x,y)] \equiv (\forall y)\,\neg[(\forall x)\,P(x,y)]$$
$$\equiv (\forall y)[(\exists x)\,\neg P(x,y)]$$

<u>Proof</u>: Let $y \in \mathbb{R}$. We must show there is $x \in \mathbb{R}$ such that $x+y \neq 1$. Take $x=-y$. Then $x+y=(-y)+y=0\neq 1$. ∎

To summarize:

$$\boxed{\begin{array}{ll} \text{(a)} & (\forall x)[(\exists y)\,P(x,y)] \\ & \text{and} \\ \text{(b)} & (\exists y)[(\forall x)\,P(x,y)] \end{array}}$$

In (a), we choose y <u>after</u> we know x.

In (b), we choose y first, and it has to work with <u>every</u> x.

Thm: Let $P(x,y)$ be a sentence depending on $x \in A$ and $y \in B$. Then

$$(\exists y \in B)(\forall x \in A)\, P(x,y) \Rightarrow (\forall x \in A)(\exists y \in B)\, P(x,y).$$

Proof: Assume $(\exists y \in B)(\forall x \in A)\, P(x,y)$ is true.

Then there is some $y_0 \in B$ such that

$$(\forall x \in A)\, P(x, y_0) \text{ is true.}$$

That is, for each $x \in A$, $P(x, y_0)$ is true.

Then $(\exists y \in B) P(x,y)$ is true for each $x \in A$, because we can take $y = y_0$.

In other words, $(\forall x \in A)(\exists y \in B) P(x,y)$ is true. ∎