Exam 2 Practice Problems

1. Let $d, n \in \mathbb{N}$. Use the definition of divisibility to show that if $d|n$, then $d \leq n$.

2. Let $a, b \in \mathbb{Z}$. Use the definition of divisibility to show that if $a|b$, then $a^2|b^2$.

3. Let a, b, q, r be integers such that $a = bq + r$. Prove that $\gcd(a, b) = \gcd(b, r)$.

4. Let $d \in \mathbb{N}$ and $n \in \mathbb{Z}$. Show that if $d|n$ and $d|(n + 1)$, then $d = 1$.

5. Use the Euclidean algorithm to compute $\gcd(84, 135)$.

6. (a) Use the Euclidean algorithm to compute $\gcd(30, 72)$.
   (b) Find integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 6$.
   (c) Do there exist integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 48$?
   (d) Do there exist integers $x, y \in \mathbb{Z}$ such that $30x + 72y = 16$?

7. Find integers x and y such that $162x + 31y = 1$.

8. Let n be an even integer. Prove that there exist unique integers $q, r \in \mathbb{Z}$ such that
$$n = 6q + r$$
and $r \in \{0, 2, 4\}$.

9. Make addition and multiplication tables for arithmetic
   (a) modulo 2.
   (b) modulo 3.
   (c) modulo 4.
   (d) modulo 5.

10. Without using a calculator, find the natural number k such that $0 \leq k \leq 14$ and k satisfies the given congruence.
    (a) $2^{75} \equiv k \mod 15$
    (b) $6^{41} \equiv k \mod 15$
    (c) $140^{874} \equiv k \mod 15$

11. Without using a calculator, show that 15 divides $37^{42} - 38^{90}$.

12. (a) Check that $r^3 \equiv r \mod 6$ for every integer r such that $0 \leq r \leq 5$.

    (b) Use part (a) to prove that $n^3 \equiv n \mod 6$ for every integer n.

    (c) If x is a real number such that $x^3 = x$, then either $x = 0$ or we can divide by x to get $x^2 = 1$ (from which we conclude $x = 1$ or $x = -1$).

    Given the result of part (b), we might wonder if similar reasoning implies that for every integer n, either $n \equiv 0 \mod 6$ or $n^2 \equiv 1 \mod 6$. Is this true?

13. Use induction to prove that

$$7^n \equiv 1 + 6n \mod 9$$

for every $n \in \mathbb{N}$.

14. (a) Let $x \in \mathbb{Z}$ and let p be a prime number. Prove that if p does not divide x, then $\gcd(p, x) = 1$.

    (b) Show that there exists $x \in \mathbb{Z}$ such that 12 does not divide x and $\gcd(12, x) \neq 1$. Why does this not contradict the result of part (a)?

15. Let P be the sentence

    For all $a, b \in \mathbb{Z}$, if $a|b$ then $a|(b + 5a^2)$.

Let Q be the sentence

    For all $a, b \in \mathbb{Z}$, if $a|b$ then $b + 5a^2$ is not prime.

    (a) Is the sentence P true? If so, provide a proof. If not, provide a counterexample.

    (b) Is the sentence Q true? If so, provide a proof. If not, provide a counterexample.

16. Use the prime factorizations

$$3{,}219{,}398 = 2 \cdot 7^3 \cdot 13 \cdot 19^2 \qquad \text{and} \qquad 158{,}184 = 2^3 \cdot 3^2 \cdot 13^3$$

to find gcd(3,219,398 , 158,184). Explain your reasoning.

17. Let $a \in \mathbb{N}$ and let p be a prime number. Prove that if $p|a^2$, then $p|a$.

[HINT: Consider the prime factorizations of a and a^2.]

18. (a) Fill in the blanks: According to the division algorithm, when we divide an integer n by 5, we obtain unique integers $q, r \in \mathbb{Z}$ such that

$$n = \underline{\hspace{3cm}}$$

and

$$\underline{\hspace{0.7cm}} \leq r \leq \underline{\hspace{0.7cm}}.$$

(b) Use the statement in part (a) to prove the following: For any integer $a \in \mathbb{Z}$, if $5|a^2$, then $5|a$.

[HINT: Apply part (a) to $n = a^2$ and to $n = a$.]

(c) Give an example of an integer a such that $8|a^2$ but $8 \nmid a$.

(d) According to part (b), the implication

$$5|a^2 \;\Rightarrow\; 5|a \tag{$\star$}$$

is **true** for every $a \in \mathbb{Z}$. But part (c) shows that the similar implication

$$8|a^2 \;\Rightarrow\; 8|a \tag{$\star\star$}$$

is **false**, at least for some integers a.

We might try to adapt the proof of ($\star$) from part (b) into a similar proof of ($\star\star$). Of course, this necessarily will fail, because ($\star\star$) is false. What, specifically, goes wrong when you do this?

19. Give examples to prove the following statements.

(a) There exist irrational numbers x and y such that $x + y$ is irrational.

(b) There exist irrational numbers x and y such that $x + y$ is rational.

(c) There exist irrational numbers x and y such that xy is irrational.

(d) There exist irrational numbers x and y such that xy is rational.

20. Let $x, y \in \mathbb{R}$. Prove the following.

   (a) If x and y are rational, then $x + y$ is rational.
   (b) If x and y are rational, then xy is rational.
   (c) If y is rational and $y \neq 0$, then $1/y$ is rational.
   (d) If x and y are rational and $y \neq 0$, then x/y is rational.
   (e) If x is rational and y is irrational, then $x + y$ is irrational.
   (f) If $x \neq 0$ is rational and y is irrational, then xy is irrational.
   (g) If y is irrational, then $1/y$ is irrational. (Why must $y \neq 0$ be true?)
   (h) If $x \neq 0$ is rational and y is irrational, then x/y is irrational.

21. Prove the following.

   [HINT: Use the fact that any rational number can be written in lowest terms.]

   (a) $\sqrt{2}$ is irrational.
   (b) $\sqrt{3}$ is irrational.
   (c) $\sqrt{6}$ is irrational.
   (d) $\sqrt{2} + \sqrt{3}$ is irrational.

22. Let's prove that $\sqrt{24}$ is irrational.

   Suppose that $a^2 = 24b^2$ (equivalently, $\left(\frac{a}{b}\right)^2 = 24$) for some $a, b \in \mathbb{N}$. We will derive a contradiction.

   (a) By the Fundamental Theorem of Arithmetic, a has a unique prime factorization. Write it as
   $$a = p_1^{e_1} \cdots p_k^{e_k}, \qquad (\star)$$
   where $p_1, \ldots, p_k$ are distinct primes (i.e., $p_i \neq p_j$ when $i \neq j$) and the exponents e_i are positive integers.

   Use $(\star)$ to write the unique prime factorization of a^2.

   (b) Describe the unique prime factorization of $24b^2$.

   [HINT: Similar to part (a), start by writing the unique prime factorization of b.]

   (c) Use the equality $a^2 = 24b^2$ and your prime factorizations from parts (a) and (b) to get a contradiction. Conclude that no such $a, b \in \mathbb{N}$ exist.