Warm-Up: Use induction to show

$$1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 1}{2}$$

for every non-negative integer n.

Parity

Def: Let $n \in \mathbb{Z}$ be an integer. Then

① n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$.

② n is **odd** if $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$

These are $\exists$ statements!

Thm: For any $x, y \in \mathbb{Z}$,

① If x is odd and y is odd, then $x+y$ is even.

② If x is even and y is odd, then $x+y$ is odd.

③ If x is even and y is even, then $x+y$ is even.

Proof: ① Suppose x and y are both odd. Then there exist integers k and l such that

$$x = 2k+1$$

and

$$y = 2l+1$$

Thus,

$$\begin{aligned} x+y &= (2k+1) + (2l+1) \\ &= 2(k+l+1). \end{aligned}$$

Since $k+l+1 \in \mathbb{Z}$, this proves $x+y$ is even.

② and ③ left as exercises. ∎

Thm: Every integer is even or odd.

Proof: Let $P(n)$ be "n is even or n is odd."

We will first prove $(\forall n \in \mathbb{N})\, P(n)$ by induction.

Base Case: When $n=1$, we have

$$1 = 2(0) + 1$$

proving that 1 is odd. So $P(1)$ is true.

Inductive Step: Let $n \in \mathbb{N}$ and assume $P(n)$ is true. That is, n is even or n is odd.

Case 1: n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Thus,

$$n+1 = 2k+1$$

is odd, proving $P(n+1)$ is true.

<u>Case 2</u>: n is odd. Then $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$. Thus,

$$\begin{aligned} n+1 = (2\ell+1)+1 &= 2\ell+2 \\ &= 2(\ell+1). \end{aligned}$$

Since $\ell + 1 \in \mathbb{Z}$, this shows $n+1$ is even. So $P(n+1)$ is true.

Thus, $P(n+1)$ is true in both cases. This completes the inductive step.

We conclude that $P(n)$ is true for each $n \in \mathbb{N}$.

It remains to prove $P(n)$ for $n \leq 0$.

<u>Zero</u>: $0 = 2(0)$ is even, so $P(0)$ is true.

Negatives: Every negative integer is of the form $-n$, where $n \in \mathbb{N}$. Thus, it suffices to prove

$$P(n) \Rightarrow P(-n)$$

for each $n \in \mathbb{N}$.

So assume $P(n)$ is true.

Case 1: n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Thus,

$$-n = -2k = 2(-k)$$

is even, since $-k \in \mathbb{Z}$.

Case 2: n is odd. Then $n = 2l+1$ for some $l \in \mathbb{Z}$. Now,

$$\begin{aligned} -n = -(2l+1) &= -2l - 1 \\ &= 2(-l-1) + 1. \end{aligned}$$

Since $-l-1 \in \mathbb{Z}$, this shows $-n$ is odd

Thus, $P(-n)$ is true in both cases.

Since $P(n)$ and $P(n) \Rightarrow P(-n)$ are both true for every $n \in \mathbb{N}$, we conclude $P(-n)$ is true for every $n \in \mathbb{N}$.

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