Warm-Up: Let $x, y \in \mathbb{Z}$. Prove that if $x+y$ is odd, then x is odd or y is odd.

Last time: Every integer is odd or even.

Is there any integer which is **both** even and odd?

This would imply 1 is even!

(Or, equivalently, $\frac{1}{2} \in \mathbb{Z}$.)

How do we know this isn't so?

Axioms for the integers

Axioms 1 – 10 on handout

- Every fact you know (or don't) about integers follows from these axioms.
- For the moment, let's imagine that we <u>only</u> know these axioms.

What can we deduce?

For example, it's not even clear that $\mathbb{N}$ is equal to $\{1, 2, 3, \ldots\}$.

Basic facts about integers

What can we deduce from the axioms?

Lemma 1: Let $a, b, c \in \mathbb{Z}$. If $a+b=a+c$, then $b=c$. [Additive Cancellation Property]

Proof: Let $a, b, c \in \mathbb{Z}$ and suppose $a+b=a+c$. Then

$$-a+(a+b)=-a+(a+c).$$

By Axiom 3 (Associativity),

$$(-a+a)+b=(-a+a)+c.$$

By Axiom 6 (Additive Inverses), we get

$$0+b=0+c,$$

and by Axiom 5 (Additive Identity), this becomes

$$b=c,$$

as desired. ∎

Lemma 2 (Uniqueness of Additive Inverses).

Let $a, b \in \mathbb{Z}$. If $a+b=0$, then $b=-a$.

Proof: Let $a, b \in \mathbb{Z}$ and suppose $a+b=0$. Since $a+(-a)=0$ also (Axiom 6), we have

$$a+b = a+(-a).$$

By the previous Lemma, we may cancel the a from both sides to get $b=-a$, as desired. ∎

Lemma 3: For any $a \in \mathbb{Z}$, $a \cdot 0 = 0$.

Proof: Let $a \in \mathbb{Z}$. Then

$$\begin{aligned} a \cdot 0 &= a(0+0) && \text{(Axiom 5)} \\ &= a \cdot 0 + a \cdot 0. && \text{(Axiom 4)} \end{aligned}$$

Also, $a \cdot 0 = a \cdot 0 + 0$ by Axiom 5. Thus,

$$a \cdot 0 + a \cdot 0 = a \cdot 0 + 0.$$

By cancellation, $a \cdot 0 = 0$. ∎