Back to our investigation of $\mathbb{Z}$:

Lemma 4: For any $a \in \mathbb{Z}$, $-(-a) = a$.

Proof: Homework 9.

Note: You may not prove this using $(-1)\cdot(-1) = 1$. Why? We will use Lemma 4 to prove $(-1)^2 = 1$.

Lemma 5: For any $a \in \mathbb{Z}$, $-a = (-1)\cdot a$.

Proof: Let $a \in \mathbb{Z}$. Since additive inverses are unique, we can prove $(-1)\cdot a = -a$ by showing that

$$a + (-1)\cdot a = 0.$$

We have

$$\begin{aligned} a + (-1)\cdot a &= (1)\cdot a + (-1)\cdot a && \text{(Axiom 5)} \\ &= (1 + (-1))\cdot a && \text{(Axiom 4)} \\ &= 0 \cdot a && \text{(Axiom 6)} \\ &= 0, && \text{(Lemma 3)} \end{aligned}$$

so $(-1)\cdot a = -a$, as desired. ∎

Now, Lemmas 4 and 5 together show that $(-1)\cdot(-1) = 1$.

Indeed,

$$\begin{aligned}(-1)\cdot(-1) &= -(-1) && \text{(Lemma 5)}\\ &= 1. && \text{(Lemma 4)}\end{aligned}$$

So far, we've only used Axioms 1-6. Axioms 7-10 concern the positive integers $\mathbb{N}$.

Note: We understand that $\mathbb{N}$ should be $\{1, 2, 3, \ldots\}$. But this isn't in the axioms, so we should refrain from assuming it to be true.

Positive Integers

Goal: Use Axioms 7-10 to show

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

Axiom 7 simply tells us that $\mathbb{N}$ exists.

↳ Define $a < b$ to mean $b - a \in \mathbb{N}$.

Axiom 8 tells us that $\mathbb{N}$ is closed under $+$ and $\cdot$

Axiom 9 tells us there is a trichotomy:

Each $a \in \mathbb{Z}$ satisfies exactly one of

- $a \in \mathbb{N} \iff a > 0$ "a is positive"
- $a = 0$
- $-a \in \mathbb{N} \iff a < 0$ "a is negative"

Axiom 10 is mysterious...

Lemma 6: $1 \in \mathbb{N}$.

Proof: By trichotomy, we only need to eliminate the other two possibilities.

- $1 \neq 0$ by Axiom 5 (Identity)
- To show $-1 \in \mathbb{N}$ is false, we will assume it is true and derive a contradiction.

Suppose $-1 \in \mathbb{N}$. Then by Axiom 8,

$$(-1)\cdot(-1) = 1 \in \mathbb{N}$$

also. But $-1 \in \mathbb{N}$ and $1 \in \mathbb{N}$ cannot both be true, by Axiom 9.

Thus, $-1 \in \mathbb{N}$ is false.

The only remaining possibility is $1 \in \mathbb{N}$. $\blacksquare$

Note: We proved that $(-1 \in \mathbb{N})$ is false by contradiction.

In general, we can prove that a sentence P is <u>false</u> as follows:

① Assume P is <u>true</u>.

② Show that this assumption leads us to a <u>contradiction</u>. That is, we are forced to conclude that a sentence Q is true, even though we already know Q to be false.

Formally, if we prove

$$P \Rightarrow Q,$$

where Q is known to be false, then P must also be false.

Lemma 7: For any $a, b \in \mathbb{Z}$, if $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Proof: Homework 9.

Note: Prove the contrapositive:

If $a \neq 0$ and $b \neq 0$, then $a \cdot b \neq 0$.

By trichotomy, $x \neq 0$ if and only if

- $x \in \mathbb{N}$ (i.e. $x > 0$)

or

- $-x \in \mathbb{N}$ (i.e. $x < 0$).

Now, consider cases.

Let's use this to prove:

Thm 8: For any $a, b, c \in \mathbb{Z}$ with $a \neq 0$, if $a \cdot b = a \cdot c$, then $b = c$.

[Multiplicative Cancellation]

Proof: Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$. Suppose $a \cdot b = a \cdot c$. Then

$$a \cdot b - a \cdot c = 0$$
$$a \cdot (b - c) = 0.$$

By the Lemma, $a = 0$ or $b - c = 0$. But $a \neq 0$, so $b - c = 0$.

That is, $b = c$. ∎

Note: No division required!

(And no division defined in $\mathbb{Z}$.)

Lemma 9: For any $a, b \in \mathbb{Z}$, exactly one of the following is true:

- $a < b$
- $a = b$
- $a > b$

[That is, $\mathbb{Z}$ is <u>linearly ordered</u> by $<$]

Proof idea: Apply trichotomy to $b-a$.

Lemma 10: Let $a, b, c \in \mathbb{Z}$.

① If $a < b$, then $a+c < b+c$

② If $a < b$ and $c > 0$, then $a \cdot c > b \cdot c$.

Proof: ① Suppose $a < b$. Then $b - a \in \mathbb{N}$.
Now,

$$(b+c)-(a+c) = b-a \in \mathbb{N},$$

so $a+c < b+c$.

(2) Suppose $a<b$ and $c>0$.
Then $b-a \in \mathbb{N}$ and $c \in \mathbb{N}$, so

$$(b-a)\cdot c \in \mathbb{N}$$

by Axiom 8. But

$$(b-a)\cdot c = b\cdot c - a\cdot c, \quad \text{(Axiom 4)}$$

so $a\cdot c < b\cdot c$.

■