Warm-Up: Let $d, n, m \in \mathbb{Z}$. Prove that if $d \mid n$ and $d \mid m$, then $d \mid (n+m)$.

Warning: $d \mid n$ is the sentence "d divides n"
d/n is the number $\frac{d}{n}$

Note: When $d \neq 0$,

$$d \mid n \text{ is true} \iff n = d \cdot k \text{ for some } k \in \mathbb{Z}$$
$$\iff \frac{n}{d} \text{ is an integer}$$

We usually avoid division, as that can take us out of the integers.

Ex:
- Every integer d divides 0, because $0 = d \cdot 0$.
- 1 divides every integer n, because $n = 1 \cdot n$.
- 0 only divides itself, because $n = 0 \cdot k \Rightarrow n = 0$.

Thm: Let $d, n \in \mathbb{Z}$. If $d \mid n$, then $(-d) \mid n$.

Proof: Suppose $d \mid n$. Then there exists $k \in \mathbb{Z}$ such that $n = dk$. Then

$$n = [(-1)\cdot(-1)]\cdot dk = (-d)(-k)$$

Since $-k \in \mathbb{Z}$, this shows $(-d) \mid n$. ∎

For this reason, we often only list positive divisors.

Ex: The divisors of 15 are $\pm 1, \pm 3, \pm 5, \pm 15$.

Thm: Let $d, n \in \mathbb{N}$. If $d | n$, then $d \leq n$.

Proof: Suppose $d | n$. Then there exists $k \in \mathbb{Z}$ such that

$$n = dk.$$

Now, $k \leq 0$ or $k \geq 1$.

Suppose, for the sake of contradiction, that $k \leq 0$. Since $d > 0$, $n = dk \leq 0$, which contradicts $n \in \mathbb{N}$.

So $k \geq 1$. Multiply by d to get

$$dk \geq d$$

i.e. $n \geq d$. $\blacksquare$

Thm: For any $a, b, c \in \mathbb{N}$,

① $a \mid a$. [Reflexivity]

② If $a \mid b$ and $b \mid a$, then $a = b$. [Antisymmetry]

③ If $a \mid b$ and $b \mid c$, then $a \mid c$. [Transitivity]

Proof: HW 10.

This theorem says divisibility is a partial order on $\mathbb{N}$.

Another partial order is $\leq$.