Real Numbers

There are two ways we could try to talk precisely about $\mathbb{R}$.

① Construct $\mathbb{R}$ from $\mathbb{Z}$

This is possible, but hard!

Step 1: Construct $\mathbb{Q}$.

- Allow division to get fractions $\frac{a}{b}$ with $a, b \in \mathbb{Z}$, $b \neq 0$.
- Impose equivalence
$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$
- Check that this is compatible with $+, \cdot$.

Step 2: Construct $\mathbb{R}$.

- Somehow use the idea that real numbers are approximated by rationals.

② Give axioms for $\mathbb{R}$

See handout.

- Most of these were also axioms for $\mathbb{Z}$
- There is no Well-Ordering axiom
- There are 2 new axioms.

⑦ Multiplicative Inverses: For each $a \in \mathbb{R}$ such that $a \neq 0$, there exists $a^{-1} \in \mathbb{R}$ such that

$$a \cdot a^{-1} = 1.$$

Write $\frac{b}{a}$ to mean $b \cdot a^{-1}$.

⑪ Least Upper Bound Property: Every non-empty subset of $\mathbb{R}$ which has an upper bound has a least upper bound in $\mathbb{R}$.

So
- everything we proved about $\mathbb{Z}$ without using Well-Ordering will also be true for $\mathbb{R}$.

- these new axioms will give $\mathbb{R}$ new properties that we did not have in $\mathbb{Z}$.

Division and Rational Numbers

Lemma: For all $a, b \in \mathbb{R}$ with $a \neq 0$ and $b \neq 0$,

(a) If $a \cdot b = 1$, then $b = a^{-1}$. [Uniqueness of Mult. Inverses]

(b) $(a^{-1})^{-1} = a$.

(c) $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$

(d) $(-a)^{-1} = -a^{-1}$

(e) $a > 0$ if and only if $a^{-1} > 0$.

In fraction Notation:
- $a \cdot b = 1 \Rightarrow b = \frac{1}{a}$
- $\frac{1}{(1/a)} = a$
- $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$
- $\frac{1}{(-a)} = -\frac{1}{a}$
- $a > 0 \Leftrightarrow \frac{1}{a} > 0$.

Proof: See handout.

Thm: Every integer is a real number.

Proof: The integers consist of positive numbers ($\mathbb{N}$), 0, and negative numbers ($-n$ for $n \in \mathbb{N}$).

$0 \in \mathbb{R}$ by Identity axiom. ✓

To show each $n \in \mathbb{N}$ is in $\mathbb{R}$, we use induction.

<u>Base Case</u>: $1 \in \mathbb{R}$ by Identity axiom.

<u>Inductive Step</u>: Let $n \in \mathbb{N}$ and suppose $n \in \mathbb{R}$. Then, since $1 \in \mathbb{R}$, we have $n+1 \in \mathbb{R}$.

✓

Lastly, since each $n \in \mathbb{N}$ is in $\mathbb{R}$, $-n$ will also be in $\mathbb{R}$ by the Additive Inverses axiom.

∎

Def: A real number $x \in \mathbb{R}$ is a <u>rational number</u> if there exist integers $a, b \in \mathbb{Z}$ such that $b \neq 0$ and $x = a \cdot b^{-1}$.

Write $x = \frac{a}{b}$, and say $\frac{a}{b}$ is a <u>fraction</u> representing x.

The set of all rational numbers is $\mathbb{Q}$.

<u>Ex</u>: $\frac{2}{3}$ and $\frac{8}{12}$ and $\frac{10}{15}$ are all different fractions representing the <u>same</u> rational number.

<u>Rule</u>: $\frac{a}{b} = \frac{c}{d} \iff a \cdot b^{-1} = c \cdot d^{-1} \iff ad = bc$

"cross-multiply"

<u>Lemma</u>: For all $x, y \in \mathbb{Q}$,

a) $x + y \in \mathbb{Q}$
b) $x - y \in \mathbb{Q}$
c) $x \cdot y \in \mathbb{Q}$
d) if $y \neq 0$, then $x \cdot y^{-1} \in \mathbb{Q}$.

<u>Proof</u>: (a) Since x and y are rational, there exist integers $a, b, c, d \in \mathbb{Z}$ such that $b \neq 0$, $d \neq 0$, and

$$x = \frac{a}{b}, \quad y = \frac{c}{d}.$$

Then

$$x + y = \frac{a}{b} + \frac{c}{d} = a \cdot b^{-1} + c \cdot d^{-1}$$

So

$$(bd) \cdot (x+y) = (bd)(ab^{-1} + cd^{-1})$$
$$= ad + bc.$$

Thus,

$$x + y = (ad + bc) \cdot (bd)^{-1}$$
$$= \frac{ad + bc}{bd}.$$

Now

- $ad + bc,\ bd \in \mathbb{Z}$
- $bd \neq 0$ because $b \neq 0$ and $d \neq 0$.

So $x + y = \frac{ad+bc}{bd} \in \mathbb{Q}$.

(b)–(d): HW 16 ■

Lemma: Let $x \in \mathbb{Q}$. Then there is $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$x = \frac{m}{n}.$$

Proof: Since x is rational, there exist $a, b \in \mathbb{Z}$ such that

$$x = \frac{a}{b}.$$

- If $b > 0$, take $m = a$ and $n = b$.
- If $b < 0$, take $m = -a$ and $n = -b$, since

$$x = \frac{a}{b} = \frac{-a}{-b}.$$

$\blacksquare$

Def: A fraction $\frac{a}{b}$ is in <u>lowest terms</u> if for every $d \in \mathbb{N}$, if $d \mid a$ and $d \mid b$, then $d = 1$.

That is, 1 is the only positive divisor a and b have in common.

Ex: $\frac{2}{3}$ is in lowest terms. $\frac{8}{12}$ is not, because $4 \mid 8$ and $4 \mid 12$.

Def: Let $x \in \mathbb{Q}$. A <u>possible positive denominator</u> for x is a positive integer $n \in \mathbb{N}$ such that there exists $m \in \mathbb{Z}$ with $x = \frac{m}{n}$.

Ex: $\frac{2}{3} = \frac{4}{6} = \frac{8}{12} = \frac{20}{30} = \cdots$

so 3, 6, 12, 30 are some of the possible denominators for this rational number.

Thm: Let $x \in \mathbb{Q}$. There exist $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $x = \frac{m}{n}$ and $\frac{m}{n}$ is in lowest terms.

Proof: Let S be the set of possible positive denominators for x.

By the lemma, x has a possible positive denominator, so S is a non-empty subset of $\mathbb{N}$.

By the Well-Ordering Principle, S has a smallest element. Call it n.

So $x = \frac{m}{n}$ for some $m \in \mathbb{Z}$.

<u>Claim</u>: $\frac{m}{n}$ is in lowest terms.

To prove this, assume it is not. Then there exists $d \in \mathbb{N}$ such that $d \mid m$ and $d \mid n$, and $d \neq 1$. So there exist $k, l \in \mathbb{Z}$ such that

$$m = dk \quad \text{and} \quad n = dl$$

Thus,

$$x = \frac{m}{n} = \frac{dk}{dl} = \frac{k}{l}.$$

Now,
- $l \in \mathbb{N}$ [because $n, d \in \mathbb{N}$]
- $l < n$ [because $d > 1$]

Thus, l is a possible positive denominator for x which is smaller than n, a contradiction. ∎

Irrational Numbers

Def: Let $x \in \mathbb{R}$. We say x is irrational if $x \notin \mathbb{Q}$.

That is, for all $a, b \in \mathbb{Z}$ with $b \neq 0$, $x \neq \frac{a}{b}$.

To show x is irrational, we assume it is rational and get a contradiction.

Thm: Let $x \in \mathbb{Q}$ and let $y \in \mathbb{R}$ be irrational.

1. $x+y$ is irrational.
2. If $x \neq 0$, then $x \cdot y$ is irrational

Proof: 1. Suppose, to get a contradiction, that $x+y \in \mathbb{Q}$. Since x is rational, $-x$ is rational (HW 16). Thus,

$$y = (x+y) + (-x)$$

is the sum of two rational numbers, so $y \in \mathbb{Q}$, a contradiction.

2. HW 16.