Warm-Up: Prove or disprove:

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers in lowest terms, then

$$\frac{ad+bc}{bd}$$

is also in lowest terms.

Let's prove that $\sqrt{2}$ is irrational. We'll use

Fact: Let $n \in \mathbb{Z}$. If n^2 is even, then n is even.
(HW 9)

Thm: For every $x \in \mathbb{Q}$, $x^2 \neq 2$.

This actually only shows $\sqrt{2} \notin \mathbb{Q}$. To prove that $\sqrt{2}$ is a real number, you need to use the Least Upper Bound Property.

Proof: Suppose, to get a contradiction, that there is some $x \in \mathbb{Q}$ such that $x^2 = 2$.

Let $x = \frac{a}{b}$ be a representation of x in lowest terms, where $a, b \in \mathbb{Z}$ and $b \neq 0$.

<u>This means</u>: If $d \in \mathbb{N}$ and $d \mid a$ and $d \mid b$, then $d = 1$.

Equivalently, $\gcd(a,b) = 1$.

We have $x^2 = \left(\frac{a}{b}\right)^2 = 2$, so $\frac{a^2}{b^2} = 2$.
Therefore,

$$a^2 = 2b^2. \qquad (*)$$

Since $b^2 \in \mathbb{Z}$, this shows a^2 is even, and thus a is even as well.

Then $a = 2k$ for some $k \in \mathbb{Z}$. Now $(*)$ becomes

$$(2k)^2 = 2b^2$$

$$4k^2 = 2b^2.$$

We may divide both sides by 2 (or use Multiplicative Cancellation) to get

$$2k^2 = b^2.$$

But this means b^2 is even, and thus so is b.

Now $2|a$ and $2|b$, which contradicts $x = \frac{a}{b}$ being in lowest terms.

We conclude that there is no such x in $\mathbb{Q}$. ∎

Ex:
- $\sqrt{3}, \sqrt{5}, \sqrt{6}$ are irrational
- $\sqrt{n}$ is irrational if $n \in \mathbb{Z}$ is not a perfect square.
- $3\sqrt{2}$ is irrational
- π and e are irrational

Hard to prove any of these!
- π Lambert, 1761
- e Euler, 1731

Here is a vast generalization:

> For each $n \in \mathbb{N}$, either
> - $\sqrt{n} \in \mathbb{N}$
>
> or
> - $\sqrt{n}$ is irrational.

This follows immediately from the following

Thm (Example 4.53 in text):
If $x \in \mathbb{Q}$ and $x^2 \in \mathbb{Z}$, then $x \in \mathbb{Z}$.

↳ Take $x = \sqrt{n}$ to prove the boxed statement.

Proof: Suppose $x \in \mathbb{Q}$ and $x^2 \in \mathbb{Z}$.

Write $x = \frac{a}{b}$ in lowest terms with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.

Our goal is to show $b=1$, so $x=a\in\mathbb{Z}$.

Let's assume $b\neq 1$ and get a contradiction.

Since $b\neq 1$ and $b\in\mathbb{N}$, we have $b>1$. Thus, there is some prime p such that $p|b$.

Now, $x^2=\frac{a^2}{b^2}=n$ for some $n\in\mathbb{Z}$. Thus,

$$a^2=b^2n=b(bn).$$

That is, $b|a^2$. By transitivity of divisibility, $p|a^2$ also.

But then $p|a$ by the "Theorem on Division by a Prime."

So $p|a$ and $p|b$, contradicting the fact that $\frac{a}{b}$ is in lowest terms.

Therefore, we conclude $b=1$ and thus $x\in\mathbb{Z}$. ∎