What about the sum of two irrational numbers?

- **It can be rational:** $\sqrt{2}$ is irrational.
  So is $-\sqrt{2} = (-1)\cdot\sqrt{2}$.
  But $\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$.

- **It can be irrational:** $\underbrace{\sqrt{2}+\sqrt{2} = 2\sqrt{2}}$ is irrational

  $\underset{\text{rational}}{2}\ \underset{\text{irrational}}{\sqrt{2}}$

The same thing happens with multiplication:

$\underset{\text{irr.}}{\sqrt{2}} \cdot \underset{\text{irr.}}{\sqrt{2}} = 2 \in \mathbb{Q}$ $\qquad$ $\underset{\text{irr.}}{\sqrt{2}} \cdot \underset{\text{irr.}}{\sqrt{3}} = \sqrt{6} \notin \mathbb{Q}$

Sets

"**Def**": A **set** is an unordered collection of objects, called **elements** of the set.

Actual definition is a list of axioms

One way to describe a set: list its elements inside braces.

Ex: $\{1,2,3\}$, $\{\text{red}, \text{blue}\}$, {☺, $, ★, □} are sets

Important notes:

- The elements in a set are **unordered**.

  So

  $$\{1,2,3\},\ \{1,3,2\},\ \{2,1,3\},\ \{2,3,1\},\ \{3,1,2\},\ \{3,2,1\}$$

  are six ways of writing **the same set**.

- The elements are **distinct** – no object can appear more than once. If we write

  $$\{1,1,1,2,2,2,2,2,2,3,3\},$$

  this means the set $\{1,2,3\}$.

Notation: If A is a set, then $x \in A$ means x is an element of A. $x \notin A$ means x is not an element of A.

Ex: $A = \{1, 2, 3\}$. Then $2 \in A$ and ☺ $\notin A$.

Sets can have sets as elements.

Ex: $A = \{ \{1,2\}, \{\text{red}, \text{blue}\}, \$ \}$ is a set with three elements, two of which are sets themselves.

$$\{1,2\} \in A$$

$$1 \notin A$$

Def: The <u>empty set</u> is the set with no elements. It is denoted $\emptyset$.

$x \in \emptyset$ is <u>false</u> for every x.

Other ways to specify sets

In words:

- Let B be the set whose elements are the first five prime numbers

  so $B = \{2, 3, 5, 7, 11\}$

- Let $\mathbb{R}_{>0}$ be the set of all positive real numbers.

By patterns:

- $E = \{2, 4, 6, 8, \ldots\}$

  (E is the set of positive even numbers)

- $P = \{2, 3, 5, 7, 11, \ldots\}$

  (P is the set of all prime numbers)

These first two methods are somewhat limited.

Set-Builder Notation: If $P(x)$ is a sentence, then

$$\{x \mid P(x)\} \quad \text{or} \quad \{x : P(x)\}$$

"such that" "such that"

is the set of all x such that $P(x)$ is true.

If A is a set, then

$$\{x \in A \mid P(x)\}$$

is the set of all x such that $x \in A$ and $P(x)$ is true. (x is a <u>bound</u> variable)

- $E = \{n \mid n \in \mathbb{N} \text{ and } n \text{ is even}\}$.

  Also, $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$.

  Also, $E = \{n \in \mathbb{N} : 2 \mid n\}$.

- $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$.

  Also, $\mathbb{R}_{>0} = \{y \in \mathbb{R} \mid y > 0\}$

<u>Transformation notation</u>: If A is a set and f is some function defined on A, then

$$\{f(x) \mid x \in A\}$$

is the set of all objects $f(x)$ obtained from all $x \in A$.

- $E = \{2n \mid n \in \mathbb{N}\}$

- $S = \{n^2 \mid n \in \mathbb{N}\}$ ← Transformation

  $= \{m \mid \text{there exists } n \in \mathbb{N} \text{ such that } n^2 = m\}$

  ↑ Set-Builder

Ex: Let $S = \{n^2 \mid n \in \mathbb{N}\}$ be the set of (positive) squares and $C = \{n^3 \mid n \in \mathbb{N}\}$ the set of (positive) cubes.

Define $A = \{x + y \mid x \in S \text{ and } y \in C\}$.
Then

<u>In words</u>: A is the set of integers which can be written as the sum of a positive square and a positive cube.

<u>Pattern</u>: $A = \{2, 5, 9, 10, 12, 17, 24, \ldots\}$

<u>Set-Builder</u>:

$A = \{n \in \mathbb{N} \mid \text{there exist } a, b \in \mathbb{N} \text{ such that } n = a^2 + b^3\}$