Subsets

Def: Let A and B be sets. We say A is a *subset* of B, written $A \subseteq B$, if $x \in A$ implies $x \in B$.

That is, $A \subseteq B$ means every element of A is also an element of B.

Can write as $(\forall x \in A)(x \in B)$
or $(\forall x)[x \in A \Rightarrow x \in B]$

Ex:
- $\{2,3,5\} \subseteq \{1,2,3,4,5\}$
- $\mathbb{N} \subseteq \mathbb{Z}$
- $\mathbb{Z} \subseteq \mathbb{Q}$
- $\mathbb{Q} \subseteq \mathbb{R}$

Ex: Any time we use set-builder notation to write

$$A = \{x \in B \mid P(x)\},$$

we have $A \subseteq B$.

Thm: For every set A, $\emptyset \subseteq A$.

Proof: The sentence $x \in \emptyset$ is always false. Thus,

$$x \in \emptyset \Rightarrow x \in A$$

is always true, so $\emptyset \subseteq A$. ∎

Thm: For every set A, $A \subseteq A$.

Proof: The sentence

$$x \in A \Rightarrow x \in A$$

is true for all x, so $A \subseteq A$. ∎

Def: Let A and B be sets. We say $\underline{A = B}$ if $A \subseteq B$ and $B \subseteq A$.

So to prove $A = B$, we usually have to prove 2 things:

- $A \subseteq B$ $\quad (x \in A \Rightarrow x \in B)$
- $B \subseteq A$ $\quad (x \in B \Rightarrow x \in A)$

Thm: Let A and B be sets. Then $A = B$ if and only if $(x \in A \Leftrightarrow x \in B)$ for all x.

Proof: $A = B$ is logically equivalent to

$$(A \subseteq B) \wedge (B \subseteq A)$$

$$\equiv (\forall x)(x \in A \Rightarrow x \in B) \wedge (\forall x)(x \in B \Rightarrow x \in A)$$

$$\equiv (\forall x)\left[(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)\right]$$

$$\equiv (\forall x)\left[x \in A \Leftrightarrow x \in B\right].$$

since $\left[(\forall x) P(x)\right] \wedge \left[(\forall x) Q(x)\right] \equiv (\forall x)\left[P(x) \wedge Q(x)\right]$

$\blacksquare$

Ex: Let's prove

$$\underbrace{\{x \in \mathbb{Z} \mid x^2 = 1\}}_{=A} = \underbrace{\{1, -1\}}_{=B}$$

We must prove $A \subseteq B$ $(x \in A \Rightarrow x \in B)$ **and** $B \subseteq A$ $(x \in B \Rightarrow x \in A)$.

Proof: $(\subseteq)$ Let $x \in A$. Then $x \in \mathbb{Z}$ and $x^2 = 1$. So

$$x^2 - 1 = 0$$
$$(x-1)(x+1) = 0.$$

Thus, $x = 1$ or $x = -1$, so $x \in B$.

$(\supseteq)$ Let $x \in B$. Then $x = 1$ or $x = -1$. Either way, $x \in \mathbb{Z}$ and $x^2 = 1$, so $x \in A$.

$\blacksquare$

Def: Let A and B be sets. We say A is a <u>proper subset</u> of B, written $A \subsetneq B$, if $A \subseteq B$ and $A = B$.

So $A \subsetneq B$ means

$$(\forall x)(x \in A \Rightarrow x \in B) \wedge (\exists y)(y \in B \wedge y \notin A).$$

Warning: Some people use $\subset$ instead of $\subseteq$. $\subset$ does <u>not</u> mean proper subset.

Warning: $\in$ vs. $\subseteq$

Ex:

$1 \in \{1,2,3\}$ is true

$\{1\} \in \{1,2,3\}$ is false

$\{1\} \subseteq \{1,2,3\}$ is true

$1 \subseteq \{1,2,3\}$ makes no sense

Ex: $\emptyset \subseteq \emptyset$ (because $\emptyset \subseteq A$ for every set A)

but $\emptyset \notin \emptyset$ (because $x \in \emptyset$ is always false)

Ex: Consider $\{\emptyset\}$, the set whose only element is $\emptyset$.

Then $\emptyset \in \{\emptyset\}$ and $\emptyset \subseteq \{\emptyset\}$.

Algebra of Sets

Def: Let A and B be sets.

① The <u>union</u> of A and B is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

② The <u>intersection</u> of A and B is the set

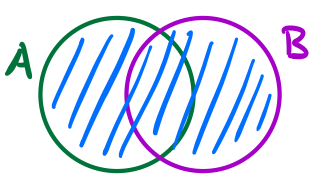
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

③ The <u>relative complement</u> of B in A is the set

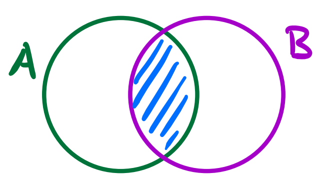
$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Also called <u>set difference</u>

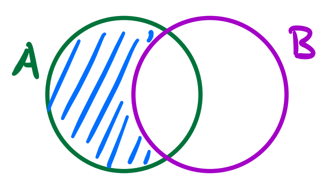
<u>Pictures:</u>



$A \cup B$



$A \cap B$



$A \setminus B$

<u>Ex</u>: Let $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$
$= \{2, 4, 6, 8, \ldots\}$

and

$P = \{p \in \mathbb{N} \mid p \text{ is prime}\}$
$= \{2, 3, 5, 7, 11, \ldots\}$

- $E \cup P = \{n \in \mathbb{N} \mid n \text{ is even or prime}\}$
  $= \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 17, \ldots\}$

- $E \cap P = \{n \in \mathbb{N} \mid n \text{ is even and prime}\}$
  $= \{2\}$

- $E \setminus P = \{4, 6, 8, 10, \ldots\}$

- $P \setminus E = \{3, 5, 7, 11, \ldots\}$

- $\mathbb{N} \setminus E = \{n \in \mathbb{N} \mid n \text{ is odd}\}$
  $= \{1, 3, 5, 7, \ldots\}$

- $E \setminus \mathbb{N} = \emptyset$