Warm-Up: List all subsets of

- $\{1\}$
- $\{1,2\}$
- $\{1,2,3\}$

Ex: Let $n \in \mathbb{N}$. A set with n elements has exactly 2^n subsets. Why?

Thm: ① For all sets A, $A \subseteq A$. [Reflexive]

② For all sets A and B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$. [Antisymmetric]

③ For all sets $A, B,$ and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. [Transitive]

Note: $\leq$ and divisibility have these same 3 properties!

Proof: ① we proved last time.

② is our definition of set equality.

③: Suppose $A \subseteq B$ and $B \subseteq C$. This means

$$x \in A \Rightarrow x \in B \quad \text{is true for every } x$$

and

$$x \in B \Rightarrow x \in C \quad \text{is true for every } x.$$

To prove $A \subseteq C$, suppose $x \in A$ for some x. Then $x \in B$ because $A \subseteq B$. Thus, $x \in C$ because $B \subseteq C$.

Therefore, $x \in A \Rightarrow x \in C$ for every x, so $A \subseteq C$.

$\blacksquare$

Many theorems from logic translate directly to theorems about sets.

Lemma: Let A and B be sets. Then for any object x,

(a) $x \notin A \cup B$ if and only if $x \notin A$ and $x \notin B$.

(b) $x \notin A \cap B$ if and only if $x \notin A$ or $x \notin B$.

Proof: (a)

$$\begin{aligned} x \notin A \cup B &\iff \neg(x \in A \cup B) \\ &\iff \neg\left[(x \in A) \vee (x \in B)\right] \\ &\iff \neg(x \in A) \wedge \neg(x \in B) \quad \text{DeMorgan} \\ &\iff (x \notin A) \wedge (x \notin B). \end{aligned}$$

(b) is similar, using the other DeMorgan Law. ∎

Thm (DeMorgan Laws for sets):

Let A, B, and S be sets. Then

(i) $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$.

(ii) $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$.

Proof: (i) We'll show both containments.

($\subseteq$): Let $x \in S \setminus (A \cup B)$. Then $x \in S$ and $x \notin A \cup B$. By the Lemma, $x \notin A$ and $x \notin B$. So $x \in S \setminus A$ and $x \in S \setminus B$. Thus, $x \in (S \setminus A) \cap (S \setminus B)$.

($\supseteq$): Let $x \in (S \setminus A) \cap (S \setminus B)$. Then $x \in S \setminus A$ and $x \in S \setminus B$. So $x \in S$ and $x \notin A$, and $x \in S$ and $x \notin B$. Since $x \notin A$ and $x \notin B$, we have $x \notin A \cup B$ by the Lemma. Thus, because $x \in S$, we have $x \in S \setminus (A \cup B)$.

(ii) is similar.

∎

Similarly, one can prove the following.

<u>Thm</u> (Commutativity of $\cup$ and $\cap$):
Let A and B be sets. Then

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$.

<u>Thm</u> (Associativity of $\cup$ and $\cap$):
Let A, B, and C be sets. Then

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$.

<u>Thm</u> (Distributive Laws for sets):
Let A, B, and S be sets. Then

(i) $S \cap (A \cup B) = (S \cap A) \cup (S \cap B)$

(ii) $S \cup (A \cap B) = (S \cup A) \cap (S \cup B)$

Sets of sets

Notation: We'll often use a script letter to denote a set of sets – i.e. a set, all of whose elements are sets.

Def: Let $\mathcal{A}$ be a set of sets. Then

- $\bigcup_{A \in \mathcal{A}} A = \{x \mid (\exists A \in \mathcal{A})(x \in A)\}$

- $\bigcap_{A \in \mathcal{A}} A = \{x \mid (\forall A \in \mathcal{A})(x \in A)\}$

Note: The book writes $\bigcup \mathcal{A}$ for $\bigcup_{A \in \mathcal{A}} A$ and $\bigcap \mathcal{A}$ for $\bigcap_{A \in \mathcal{A}} A$.

Ex: Let $\mathcal{A} = \{\{1,2\}, \{2,3\}, \{2,5,6\}\}$. Then

$$\bigcup_{A \in \mathcal{A}} A = \{1,2\} \cup \{2,3\} \cup \{2,5,6\} = \{1,2,3,5,6\}$$

and

$$\bigcap_{A \in \mathcal{A}} A = \{1,2\} \cap \{2,3\} \cap \{2,5,6\} = \{2\}.$$

Ex: Let $A_n = \{k \in \mathbb{N} \mid k \geq n\}$
$= \{n, n+1, n+2, \ldots\}$

So $A_1 = \{1, 2, 3, \ldots\} = \mathbb{N}$
$A_2 = \{2, 3, 4, \ldots\}$
$A_3 = \{3, 4, 5 \ldots\}$
$\vdots$

Set $\mathcal{A} = \{A_n \mid n \in \mathbb{N}\}$
$= \{A_1, A_2, A_3, \ldots\}$.

A set with infinitely many elements, each of which is a set

Then

$$\bigcup_{A \in \mathcal{A}} A = \bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \cdots = \mathbb{N}.$$

Proof: Let $x \in \bigcup_{n=1}^{\infty} A_n$. Then $x \in A_n$ for some n. But $A_n \subseteq \mathbb{N}$, so $x \in \mathbb{N}$. Thus, $\bigcup_{i=1}^{\infty} A_i \subseteq \mathbb{N}$.

On the other hand, let $x \in \mathbb{N}$. Since $\mathbb{N} = A_1$, $x \in \bigcup_{n=1}^{\infty} A_n$. Thus, $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$. ■

Also,

$$\bigcap_{A \in \mathcal{A}} A = \bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots = \emptyset.$$

Proof: Suppose $x \in \bigcap_{n=1}^{\infty} A_n$. Then $x \in A_n$ for every n. In particular, $x \in A_1 = \mathbb{N}$. But then $x \notin A_{x+1}$, which contradicts $x \in A_n$ for all $n \in \mathbb{N}$.

So $\bigcap_{n=1}^{\infty} A_n$ must be empty. ■