Functions

Def: Let A and B be sets. A function $f: A \to B$ is a rule which associates to each $x \in A$ an element $f(x) \in B$.

- A is the domain of f, written $A = \mathrm{Dom}(f)$.
  (the set of all valid inputs)
  We might say f is a function on A.

- B is the target or codomain of f.
  (a set containing all possible outputs)

- For $x \in A$, $f(x)$ is the value of f at x.
  [f is the function, $f(x)$ is an element of B]

- The word map is a synonym for function.

Note that to define a function, we must specify both the domain and the target.

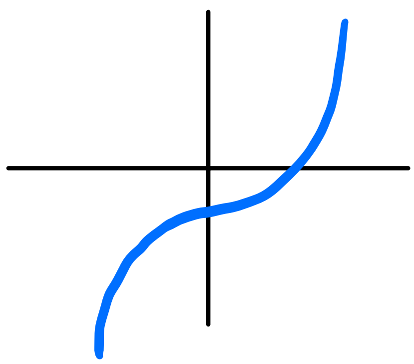
Graphs

Def: Let $f: A \to B$ be a function.
The graph of f is

$$\mathrm{Graph}(f) = \{(x,y) \in A \times B \mid y = f(x)\}.$$

Observe: For each $x \in A$, there is a unique $y \in B$ such that $(x,y) \in \mathrm{Graph}(f)$, namely $y = f(x)$.
"vertical line test"

Ex: For $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 2$, the graph of f is



What is this? It's

$$\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid y = x^3 - 2\}.$$

Aside: You can actually use graphs to define what a function is.

Def: Let A and B be sets. A function $f: A \to B$ is a subset

$$\mathrm{Graph}(f) \subseteq A \times B$$

with the property that for all $x \in A$, there exists a unique $y \in B$ such that $(x, y) \in \mathrm{Graph}(f)$.

If $(x,y) \in \mathrm{Graph}(f)$, write $f(x) = y$.

Note:
- You don't have to use this definition. But it's more concrete than defining a function as a "rule".
- We can't always draw $\mathrm{Graph}(f)$.

Def: Let $f: A \to B$ be a function. The <u>range</u> of f, denoted $\mathrm{Rng}(f)$, is the set

$$\mathrm{Rng}(f) = \{y \in B \mid f(x) = y \text{ for some } x \in A\}.$$

Note: $\mathrm{Rng}(f) \subseteq B$ automatically.

Informally, $\mathrm{Rng}(f)$ is the set of all function values.

Equivalently,

$$\mathrm{Rng}(f) = \{y \in B \mid (x,y) \in \mathrm{Graph}(f)\}$$

Ex: Let $A = \{a, b, c, d\}$. Define $f: A \to \mathbb{Z}$ by

$$f(a) = 2, \quad f(b) = 3, \quad f(c) = 1, \quad f(d) = 1.$$

When the domain is finite, like it is here, we can represent the function as a table.

Dummy variable

x	$f(x)$
a	2
b	3
c	1
d	1

$$\mathrm{Rng}(f) = \{1, 2, 3\} \subseteq \mathbb{Z}.$$

Ex: Consider the functions

Dummy variable

$f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

$g: \mathbb{R} \to [0,\infty)$ " " $g(x) = x^2$

$h: \mathbb{R} \to [-2,\infty)$ " " $h(x) = x^2$

$i: [0,\infty) \to [0,\infty)$ " " $i(x) = x^2$

$j: [1,2] \to \mathbb{R}$ " " $j(x) = x^2$

$k: [1,2] \to [1,4]$ " " $k(x) = x^2$

Q: Why can't we add
$\ell: [1,3] \to [1,5]$ given by $\ell(x) = x^2$
to this list?

Def: We say two functions f and g are <u>equal</u> if

① $\mathrm{Dom}(f) = \mathrm{Dom}(g)$

and

② For every $x \in \mathrm{Dom}(f)$, $f(x) = g(x)$.

↑ equality of values

In this case we write $f = g$.

↑ equality of functions

$$f = g \iff \mathrm{Graph}(f) = \mathrm{Graph}(g)$$

Ex: In the previous example, we have 3 functions up to equality: $f = g = h$, i, and $j = k$.

Ex: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Then $\operatorname{Rng}(f) = [0, \infty)$.

Proof: $(\subseteq)$ Let $y \in \operatorname{Rng}(f)$. Then $y \in \mathbb{R}$ and $y = f(x)$ for some $x \in \mathbb{R}$. Thus $y = x^2 \geq 0$, so $y \in [0, \infty)$.

$(\supseteq)$ Let $y \in [0, \infty)$. Then $y \geq 0$, so $\sqrt{y} \in \mathbb{R}$. Set $x \in \sqrt{y}$. We have

$$f(x) = x^2 = (\sqrt{y})^2 = y,$$

which shows that $y \in [0, \infty)$. ∎

Function Composition

Def: Let $f: A \to B$ and $g: B \to C$ be functions. The composition of g with f is the function

$$g \circ f : A \to C$$

given by

$$(g \circ f)(a) = g(f(a))$$

"g after f"

for all $a \in A$.

Ex: $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2^x$
$g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$.

Then $(g \circ f)(x) = (2^x)^2 = 2^{2x} = 4^x$

and

$$(f \circ g)(x) = 2^{(x^2)}$$

$g \circ f \neq f \circ g$, since $(g \circ f)(1) = 4$
$(f \circ g)(1) = 2$

Order matters!

Note: Read compositions from right to left

- Sometimes, $g \circ f$ is defined but $f \circ g$ is not

Picture:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Thm: Let $f: A \to B$, $g: B \to C$, and $h: C \to D$ be functions. Then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Proof idea: Both are given by $x \mapsto h(g(f(x)))$.