Warm-Up: Prove that

$$f: \mathbb{R}\setminus\{3\} \to \mathbb{R}\setminus\{1\}$$
$$x \mapsto \frac{x}{x-3}$$

is a bijection.

Inverse Functions

A bijection $f: A \to B$ gives us a rule for going back to B from A. Specifically, $y \in B$ can map back to the unique $x \in A$ such that $f(x) = y$.

Def: Let $f: A \to B$ be a bijection. The inverse function of f is

$$f^{-1}: B \to A$$

defined as follows: For each $y \in B$, $f^{-1}(y)$ is the unique element $x \in A$ such that $f(x) = y$.

That is $f^{-1}(y) = x \iff y = f(x)$.

Ex: $f: \mathbb{R} \to (0,\infty)$ given by $f(x) = e^x$ is a bijection.

$f^{-1}: (0,\infty) \to \mathbb{R}$ is given by $f^{-1}(y) = \ln(y)$.

$$\ln(y) = x \iff y = e^x$$

Ex: $g: [0,\infty) \to [0,\infty)$ is a bijection.
$x \longmapsto x^2$

Its inverse is $g^{-1}: [0,\infty) \to [0,\infty)$
$y \longmapsto \sqrt{y}$

$$\sqrt{y} = x \iff y = x^2 \text{ and } x \geq 0$$

Ex: $\sin: \mathbb{R} \to \mathbb{R}$ is not a bijection,
but $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [1,1]$ is.

Its inverse is $\sin^{-1}: [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\sin^{-1}(y) = x \iff y = \sin(x) \text{ and } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

Def: Let S be any set. The <u>identity function</u> on S is

$$\begin{aligned} id_S : S &\to S. \\ x &\mapsto x \end{aligned}$$

That is, $id_S(x) = x$ for all $x \in S$.

Observe: $id_S : S \to S$ is a bijection.

Ex: If $f : A \to S$ and $g : S \to B$ are functions, then

$$id_S \circ f = f \quad \text{and} \quad g \circ id_S = g.$$

That is, id_S is the identity with respect to function composition.