Warm-Up: Prove that the function

$$f : \mathbb{N} \longrightarrow \mathbb{N} \setminus \{1\}$$
$$x \longmapsto x+1$$

is a bijection.

Thm: Let $f : A \to B$ be a bijection and let $f^{-1} : B \to A$ be its inverse. Then

① $f^{-1} \circ f = id_A : A \to A$

and

② $f \circ f^{-1} = id_B : B \to B$

This is essentially a rephrasing of the fundamental identity $f^{-1}(y) = x \iff f(x) = y$.

Proof: ① Let $x \in A$. We must show

$$(f^{-1} \circ f)(x) = id_A(x) = x.$$

Set $y = f(x)$. Then, by definition of f^{-1}, $f^{-1}(y) = x$. But then

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x. \quad ✓$$

② Let $y \in B$. We must show

$$(f \circ f^{-1})(y) = id_B(y) = y.$$

Set $x = f^{-1}(y)$. Then $f(x) = y$, so

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y.$$ ✓

Cor: Let $f: A \to B$ be a bijection. Then its inverse $f^{-1}: B \to A$ is also a bijection, and $(f^{-1})^{-1} = f$.

Proof: Let $f: A \to B$ be a bijection.

- f^{-1} is surjective: Let $x \in A$.

We must find $y \in B$ so that $f^{-1}(y) = x$.

Set $y = f(x)$. Then, by the theorem,

$$f^{-1}(y) = f^{-1}(f(x)) = x.$$ ✓

- **f^{-1} is injective**: Let $y_1, y_2 \in B$ such that $f^{-1}(y_1) = f^{-1}(y_2)$.
Then
$$f(f^{-1}(y_1)) = f(f^{-1}(y_2)),$$
so by the theorem,
$$y_1 = y_2.$$
✓

- **$(f^{-1})^{-1} = f$**: By definition, for $x \in A$ and $y \in B$,
$$(f^{-1})^{-1}(x) = y \iff x = f^{-1}(y) \iff f(x) = y.$$
Thus, $(f^{-1})^{-1} = f$. $\blacksquare$

The following theorems are proved using similar methods.

Thm: Let $f: A \to B$ and $g: B \to A$ be functions.
If
$$g \circ f = id_A \quad \text{and} \quad f \circ g = id_B,$$
then f is a bijection and $g = f^{-1}$.

Thm: If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection also, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Cardinality

What does it mean for a set A to have exactly n elements?

Ex: $A = \{4, \text{red}, \$\}$ has exactly 3 elements

How do we know? We can list them:

1. 4
2. red
3. \$

This is just a bijection $f: \{1, 2, 3\} \to A$.

surjection $\iff$ every element in A is on the list

injection $\iff$ no element in A is on the list more than once.

Def: Let A and B be sets. We say A and B <u>have the same cardinality</u>, denoted $|A| = |B|$, if there exists a bijection $f : A \to B$.

<u>Book:</u> A and B are <u>equinumerous</u>, $\bar{\bar{A}} = \bar{\bar{B}}$.

If A is a set and $n \in \mathbb{N}$ such that A and $\{1, 2, \dots, n\}$ have the same cardinality, then we say A has <u>cardinality n</u> (or A has <u>exactly n elements</u>), and write $|A| = n$.

We also write $|\emptyset| = 0$.

This is an equivalence relation.

Thm: Let A, B, C be sets. Then

① $|A| = |A|$. [Reflexive]

② If $|A| = |B|$, then $|B| = |A|$. [Symmetric]

③ If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.
[Transitive]

Proof sketch: ① $\mathrm{id}_A : A \to A$ is a bijection.
$x \mapsto x$

② If $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is a bijection.

③ If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection. (HW 22)