Finite and infinite sets

Def: A set A is *finite* if

- $A = \emptyset$ $\iff |A| = 0$

or

- For some $n \in \mathbb{N}$, there is a bijection $f : \{1, 2, \ldots, n\} \to A$. $\iff |A| = n$

Think: f lists all elements of A on n lines with no repeats.

Ex: $A = \{4, \text{red}, \$\}$. $|A| = 3$

Ex: $B = \{a, b, c, \ldots, z\}$. $|B| = 26$

Warm-Up: You probably intuit that $\mathbb{N}$ is infinite (i.e., not finite).

Try to prove this.

Hint: Use contradiction.

Here's one solution: Suppose $|\mathbb{N}| = n$, so there is a bijection $f: \{1, 2, \ldots, n\} \to \mathbb{N}$. Let

$$M = \text{maximum of } f(1), f(2), \ldots, f(n).$$

Then $f(k) \leq M < M+1$ for all $k \in \{1, \ldots, n\}$, so $M+1 \in \mathbb{N} \setminus \operatorname{Rng}(f)$.

Thus, f is not surjective, so cannot be a bijection. ∎

Ex: Similarly, $\mathbb{Q}$ and $\mathbb{R}$ are infinite.

WARNING: It may be tempting to write

$$|\mathbb{N}| = \infty$$
$$|\mathbb{Q}| = \infty$$
$$|\mathbb{R}| = \infty$$

We will not do this.

As we will see, $|\mathbb{N}| = |\mathbb{Q}|$, but $|\mathbb{N}| \neq |\mathbb{R}|$.

First, more on finite sets.

Ex: Is there a set C such that $|C|=3$ and $|C|=26$? No!

Thm: Let $n, m \in \mathbb{N} \cup \{0\}$. If A is a set such that $|A|=n$ and $|A|=m$, then $n=m$.

Proof idea:

$|A| = n$ means there is a bijection $f:\{1,\dots,n\} \to A$

$|A| = m$ " " " $g:\{1,\dots,m\} \to A$.

So $g^{-1} \circ f : \{1,\dots,n\} \to \{1,\dots,m\}$ is a bijection.

- If $n > m$, this can't be injective.
- If $n < m$, this can't be surjective.

Thm: Let S be a finite set and $T \subseteq S$. Then

- T is finite
- $|T| \leq |S|$
- $|T| = |S|$ if and only if $T = S$.

Proof: Book Thms 13.30, 13.33.

(By induction on $|S|$. Not hard - just tedious.)

Cor: Let A and B be finite sets, and let $f: A \to B$ be a function. Then

① If f is an injection, then $|A| \leq |B|$
② If f is a surjection, then $|A| \geq |B|$

Proof: ① Suppose $f: A \to B$ is injective. Then
$$f: A \to \text{Rng}(f)$$
is a bijection. Hence, $|A| = |\text{Rng}(f)|$.
But $\text{Rng}(f) \subseteq B$, so $|\text{Rng}(f)| \leq |B|$ by the Theorem. Together, we get $|A| \leq |B|$.

② Suppose $f: A \to B$ is surjective. Since B is finite, $|B| = n$ for some $n \in \mathbb{N}$, so we can write
$$B = \{b_1, b_2, \ldots, b_n\}.$$
For each $i \in \{1, \ldots, n\}$, let $a_i \in A$ be such that $f(a_i) = b_i$.

If $i \neq j$, then $f(a_i) = b_i \neq b_j = f(a_j)$, so $a_i \neq a_j$.

Thus, $|\{a_1, \dots, a_n\}| = n$. But $\{a_1, \dots, a_n\} \subseteq A$, so $n \leq |A|$. Since $|B| = n$, we have $|A| \geq |B|$.

$\blacksquare$

The contrapositive of ① is the

Pigeonhole Principle: Let A and B be finite sets and $f: A \to B$ a function. If $|A| > |B|$, then f is not injective.

A - set of pigeons

B - set of pigeonholes

$f: A \to B$ puts each pigeon in a pigeonhole

Then there is a pigeonhole containing more than one pigeon.

Ex: If $a_1, a_2, a_3, a_4 \in \mathbb{Z}$, then the difference $a_i - a_j$ will be divisible by 3 for some $i \neq j$.

Ex: Suppose n people are at a party. Then there are two people who have the same number of friends at the party.

↳ Cannot be someone with 0 friends and someone with $n-1$ friends. So possibilities are $0, \dots, n-2$ or $1, \dots, n-1$.