Warm-Up: Use a truth table to show that $P \Rightarrow Q$ is **not** logically equivalent to $Q \Rightarrow P$.

A sentence of the form $P \Rightarrow Q$ is called a **conditional sentence**.

Ways to say $P \Rightarrow Q$:

"P implies Q"
"If P, then Q"
"P is sufficient for Q"
"Q is necessary for P"

In a conditional sentence $P \Rightarrow Q$, P is the **antecedent** and Q is the **consequent**.

More informally, P is the "assumption" and Q is the "conclusion."

Converse and contrapositive

Let P and Q be sentences.

The <u>converse</u> of $P \Rightarrow Q$ is the sentence

$$Q \Rightarrow P.$$

The <u>contrapositive</u> of $P \Rightarrow Q$ is the sentence

$$\neg Q \Rightarrow \neg P.$$

<u>Ex</u>: "If it is raining, then the ground is wet."

Converse: "If the ground is wet, then it is raining."

Contrapositive: "If the ground is dry, then it is not raining."

We saw in the Warm-Up that $P \Rightarrow Q$ is <u>not</u> logically equivalent to the converse $Q \Rightarrow P$.

<u>Thm</u>: $P \Rightarrow Q$ is logically equivalent to the contrapositive $\neg Q \Rightarrow \neg P$.

<u>Proof</u>: We have

$$\begin{aligned} \neg Q \Rightarrow \neg P &\equiv \neg(\neg Q) \vee \neg P \\ &\equiv Q \vee \neg P \\ &\equiv \neg P \vee Q \\ &\equiv P \Rightarrow Q. \end{aligned}$$

$\blacksquare$

Alternatively:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

A final logical connective:

(5) Biconditional: $\Leftrightarrow$ means "if and only if"

$P \Leftrightarrow Q$ is true exactly when P and Q have the same truth value.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Thm: $P \Leftrightarrow Q$ is logically equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$.

Proof:

P	Q	$P \Leftrightarrow Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

$\blacksquare$

A sentence of the form $P \Leftrightarrow Q$ is called a <u>biconditional sentence</u>.

Ways to say $P \Leftrightarrow Q$:

"P if and only if Q"

"P is necessary and sufficient for Q"

"Q is necessary and sufficient for P"

"P is necessary for Q" is $Q \Rightarrow P$

"P is sufficient for Q" is $P \Rightarrow Q$

<u>Ex</u>: $x^2 = 9 \iff x = 3$ or $x = -3$

This sentence is true. Why?

Let P be "$x^2 = 9$" and Q be "$x = 3$ or $x = -3$".

We'll show $P \Rightarrow Q$ and $Q \Rightarrow P$ are both true.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$P \Rightarrow Q$

Case 1: P is true. Then $x^2 = 9$, so

$$x^2 - 9 = 0.$$

Factor to get $(x-3)(x+3) = 0$.

Hence, $x - 3 = 0$ or $x + 3 = 0$.

That is, $x = 3$ or $x = -3$, so Q is true. ✓

Case 2: P is false. Then $P \Rightarrow Q$ is vacuously true. ✓

$Q \Rightarrow P$

Case 1: Q is true. Then $x = 3$ or $x = -3$, so

$$x^2 = 3^2 = 9 \quad \text{or} \quad x^2 = (-3)^2 = 9.$$

That is, P is true. ✓

Case 2: Q is false. Then $Q \Rightarrow P$ is vacuously true. ✓

Conditional Proof

In general, to show $P \Rightarrow Q$ is true, we must

1. Assume P is true.
2. Under this assumption, show that Q must be true also.

Why is this valid?

When P is false, $P \Rightarrow Q$ is automatically true.

This method is called <u>conditional proof</u>.

Most of our theorems will be of the form $P \Rightarrow Q$, so we will write a lot of conditional proofs.

To prove a biconditional $P \Leftrightarrow Q$, we need two conditional proofs: for $P \Rightarrow Q$ and $Q \Rightarrow P$.