Compare to HW 3 #1

Warm-Up: Let x and y be real numbers.

Let S be the conditional sentence

"If $xy > 0$, then $x > 0$ and $y > 0$."

- Is S true or false?
- Write the converse and contrapositive of S. Are these sentences true or false?
- Write the negation $\neg S$ as an "and" sentence.

Another perspective on conditional proof:

To prove $A \Rightarrow C$ is true, it is enough to show that it is impossible for A to be true and C to be simultaneously false.

To do this, either

- Assume A is true and show C must also be true.

OR

- Assume C is false and show A must also be false. [i.e. prove $\neg C \Rightarrow \neg A$]

Tautologies

A sentence is called a tautology if it is always true for structural reasons (i.e. because of how it is constructed using $\neg$, $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$).

Ex: "Modus ponens"

$$(P \Rightarrow Q) \wedge P \Longrightarrow Q$$

is a tautology.

Proof: Assume $(P \Rightarrow Q) \wedge P$ is true.

[Again, if it's false there's nothing to do.]

Then both of $P \Rightarrow Q$ and P are true, so Q must be true. ∎

Modus ponens is a common step in logical reasoning.

Ex: If it is raining, then the ground is wet. It is raining.

Therefore, the ground is wet.

Ex: $P \vee \neg P$ is a tautology.

"The law of the excluded middle."

Ex: Show that the sentence

$$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$$

is a tautology. "Hypothetical syllogism"

Proof: Suppose $(P \Rightarrow Q) \wedge (Q \Rightarrow R)$ is true. Thus, $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true.

We want to argue that $P \Rightarrow R$ must be true.

Assume P is true. [If not, $P \Rightarrow Q$ is true vacuously and we are done.]

Since $P \Rightarrow Q$ and P are both true, Q must be true.

Since $Q \Rightarrow R$ and Q are both true, R must be true.

Therefore, $P \Rightarrow R$ is true, as desired. $\blacksquare$

Revisiting the Warm-Up:

<u>Recall</u>: The truth value of the sentence "If $xy > 0$, then $x > 0$ and $y > 0$" depends on the numerical values of x and y.

We said it is false because it <u>can</u> be false (e.g. $x = -1$, $y = -1$), but it can also be true (e.g. $x = 1$, $y = 1$ or $x = -1$, $y = 1$).

<u>Quantifiers</u> let us discuss such situations.

Quantifiers

The <u>universal quantifier</u> is $\forall$, which means "for all."

If $P(x)$ is a sentence involving the variable x, then $(\forall x)\, P(x)$ is the sentence

"for all x, $P(x)$"

also read as

"for every x, $P(x)$"
"for each x, $P(x)$"
"for any x, $P(x)$."

The <u>existential quantifier</u> is $\exists$, which means "there exists."

$(\exists x)\, P(x)$ is the sentence

"there exists x such that $P(x)$"

also read as

"for some x, $P(x)$"

"for at least one x, $P(x)$."

<u>Note</u>:
- A quantifier $(\forall, \exists)$ is always "attached" to a variable, called the <u>bound variable</u>.
- A quantifier is always followed by a sentence involving the bound variable.