Warm-Up: Show that the sentence

$$(P \land Q) \Rightarrow P$$

is a tautology – that is, it is true regardless of the sentences P and Q.

Do this ① Using a truth table.

② By writing a conditional proof.

Last time: We introduced the quantifiers

$\forall$ "for all"

$\exists$ "there exists"

Usage is

$$(\underbrace{\overset{\text{quantifier}}{\forall} x}_{\text{bound variable}})\ \overset{\text{sentence involving the bound variable}}{P(x)}$$

Ex: Let $P(x)$ be the sentence

$$\text{“}(x > 1) \Rightarrow (x^2 > 1)\text{”}$$

and let $Q(x)$ be the converse

$$\text{“}(x^2 > 1) \Rightarrow (x > 1).\text{”}$$

Then
- $(\forall x)\, P(x)$ is true.
- $(\exists x)\, P(x)$ is true.
- $(\forall x)\, Q(x)$ is false.
- $(\exists x)\, Q(x)$ is true.

Note: We should be more careful to specify which values the bound variable is allowed to take on.

The above statements are correct when x can be any real number.

To indicate this, we will write $(\forall x \in \mathbb{R})$ and $(\exists x \in \mathbb{R})$.

When we use a quantifier ($\forall$ or $\exists$), a bound variable is ranging over a <u>universe</u> of possibilities.

Usually, we should be explicit about this.

<u>Common choices</u>:

$\mathbb{Z}$ = the set of integers
$\mathbb{Q}$ = the set of rational numbers
$\mathbb{R}$ = the set of real numbers
$\mathbb{C}$ = the set of complex numbers

The universe matters!

<u>Ex</u>: $(\exists x)(x^2 = 2)$ Ambiguous

$(\exists x \in \mathbb{Z})(x^2 = 2)$ <u>False</u>, $\sqrt{2} \notin \mathbb{Z}$

$(\exists x \in \mathbb{R})(x^2 = 2)$ <u>True</u>, $\sqrt{2} \in \mathbb{R}$

<u>Ex</u>: $(\forall x \in \mathbb{R})(x^2 \geq 0)$ <u>True</u>

$(\forall x \in \mathbb{C})(x^2 \geq 0)$ <u>False</u>, $\sqrt{-1} \in \mathbb{C}$

Ex: Which statements are true?

① $(\exists x \in \mathbb{R})(x+4=9)$

True: $x=5$.

② $(\forall x \in \mathbb{R})(x+4=9)$

False: Try $x=0$.

③ $(\exists x \in \mathbb{R})[(x+4=9) \wedge (x \neq 5)]$

False: $x+4=9 \Rightarrow x=9-4=5$

④ $(\exists x \in \mathbb{R})(x^2+6x+8 \geq 0)$

True: Try $x=0$.

⑤ $(\forall x \in \mathbb{R})(x^2+6x+8 \geq 0)$

Can guess and check, or complete the square:

$$x^2+6x+8 = x^2+6x+9-1$$
$$= (x+3)^2-1.$$

False: Try $x=-3$.

⑥ $(\forall x \in \mathbb{R})(x^2 + 6x + 10 \geq 0)$

<u>True</u>: $x^2 + 6x + 10 = (x+3)^2 + 1 \geq 1 > 0$ for all real numbers x.

<u>Observe</u>:

- A single example proves a $\exists$ statement.
- A single counterexample disproves a $\forall$ statement.
- To prove a $\forall$ statement or disprove a $\exists$ statement, we need an argument that works for <u>all</u> values.

Note: Over a finite set (universe),

- $\forall$ is an "and" statement
- $\exists$ is an "or" statement

Ex: If $A = \{-3, 1, 4\}$, then

$$(\forall x \in A)(x^2 < 20) \equiv ((-3)^2 < 20) \wedge (1^2 < 20) \wedge (4^2 < 20)$$

$$(\exists x \in A)(x > 0) \equiv (-3 > 0) \vee (1 > 0) \vee (4 > 0)$$

(Both true)

For this reason, we can think of $\forall$ as "generalized and" and $\exists$ as "generalized or."