Warm-Up: For each sentence, draw a number line and indicate all x-values making the sentence true.

(a) $(x > 2) \wedge (x^2 > 4)$

(b) $(x > 2) \vee (x^2 > 4)$

(c) $(x > 2) \Rightarrow (x^2 > 4)$

(d) $(x > 2) \Leftrightarrow (x^2 > 4)$

Free + Bound Variables

Let $P(x) =$ "$x^2 + 6x + 8 \geq 0$."

- Is $P(x)$ true? It depends on x.

We say that x is a free variable in the sentence $P(x)$.

Think: The sentence $P(x)$ is a function of x.

- Is $(\forall x \in \mathbb{R})\, P(x)$ true? No!

  This sentence does NOT depend on x, because of the quantifier $\forall$.

  In this case, we say x is a <u>bound</u> <u>variable</u> in the sentence $(\forall x \in \mathbb{R})\, P(x)$.

The quantifier $\exists$ can also bound variables: $(\exists x)\, P(x)$ does not depend on x.

<u>Analogy</u>: $f(x) = x^2$ vs. $\int_0^1 x^2 \, dx$

Thm (Generalized DeMorgan's Laws)

(a) $\neg[(\forall x \in A)\ P(x)] \equiv (\exists x \in A)(\neg P(x))$

(b) $\neg[(\exists x \in A)\ P(x)] \equiv (\forall x \in A)(\neg P(x))$

Proof: (a) Suppose $\neg[(\forall x \in A)\ P(x)]$ is true.
Then $(\forall x \in A)\ P(x)$ is false.

So there is some $x_0 \in A$ such that $P(x_0)$ is false, i.e. $\neg P(x_0)$ is true.

Hence $(\exists x \in A)(\neg P(x))$ is true.

Conversely, suppose $(\exists x \in A)(\neg P(x))$ is true.

Then there is $x_0 \in A$ such that $\neg P(x_0)$ is true, i.e. $P(x_0)$ is false.

So $(\forall x \in A)\ P(x)$ is false. Therefore, $\neg(\forall x \in A)\ P(x)$ is true.

(b) is similar (see book). $\blacksquare$

Thm (Generalized Distributive Laws):
Let P be a sentence not involving x.
Let $Q(x)$ be a sentence involving x.

Then
a) $P \wedge [(\exists x \in A)\, Q(x)] \equiv (\exists x \in A)\, [P \wedge Q(x)]$
b) $P \vee [(\forall x \in A)\, Q(x)] \equiv (\forall x \in A)\, [P \vee Q(x)]$.

Proof: Omitted (see book).

Order of Quantifiers

Suppose $P(x,y)$ is a sentence involving $x \in A$ and $y \in B$.

What is the difference between

① $(\forall x \in A) [(\exists y \in B)\, P(x,y)]$

② $(\exists y \in B) [(\forall x \in A)\, P(x,y)]$?

implicit

Ex: $A = B = \mathbb{R}$, $\quad P(x,y) =$ "$x+y=1$"

① is "for every $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x+y=1$."

True!

<u>Proof</u>: Let $x \in \mathbb{R}$. Set $y = 1-x \in \mathbb{R}$. Then $x+y = x+(1-x) = 1$. ∎

② is "there exists some $y \in \mathbb{R}$ such that for every $x \in \mathbb{R}$, we have $x+y=1$."

False!

<u>Proof</u>: Suppose $y \in \mathbb{R}$. Then $x+y=1$ is <u>not</u> true for <u>every</u> $x \in \mathbb{R}$, since we could take $x=-y$ and get $x+y = (-y)+y = 0 \neq 1$. ∎

By Generalized DeMorgan, the negation

$$\neg(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x+y=1) \equiv (\forall y \in \mathbb{R})\neg(\forall x \in \mathbb{R})(x+y=1)$$
$$\equiv (\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(x+y \neq 1)$$

is <u>True</u>, and this is exactly what we proved.

To summarize:

$$\boxed{\begin{array}{l} ① \ (\forall x \in A)[(\exists y \in B)\ P(x,y)] \\ ② \ (\exists y \in B)[(\forall x \in A)\ P(x,y)] \end{array}}$$

In ①, we choose y after we know x.

In ②, we choose y first, and it has to work with every x.

Thm: Let $P(x,y)$ be a sentence depending on $x \in A$ and $y \in B$. Then

$$\underbrace{(\exists y \in B)(\forall x \in A) P(x,y)}_{②} \Rightarrow \underbrace{(\forall x \in A)(\exists y \in B) P(x,y)}_{①}$$

Proof: Assume $(\exists y \in B)(\forall x \in A) P(x,y)$ is true.

Then there is some y-value $y_0 \in B$ such that $(\forall x \in A) P(x, y_0)$ is true.

That is, $P(x, y_0)$ is true for each $x \in A$.

Thus, for any $x \in A$, the sentence $(\exists y \in B) P(x,y)$ is true, because we can take $y = y_0$.

In other words, $(\forall x \in A)(\exists y \in B)P(x,y)$ is true, as desired. $\blacksquare$

Ex: $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(xy = 0)$

True: Take $y = 0$.

Thus, $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 0)$ is also true.