3 Definitions

- surjective (onto)
- injective (one-to-one)
- bijective (both - invertible)

Surjections

<u>Def</u>: Let $f: A \to B$ be a function. We say f is a <u>surjection</u> if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Also say f is <u>surjective</u>, f is <u>onto</u>.

Equivalently, $f: A \to B$ is surjective $\iff \underbrace{\mathrm{Rng}(f)}_{= \{y \in B \mid y = f(x) \text{ for some } x \in A\}} = B$

Injections

Def: A function $f: A \to B$ is an injection if for all $x_1, x_2 \in A$,

if $f(x_1) = f(x_2)$, then $x_1 = x_2$

Also, say f is injective or f is one-to-one.

Equivalently, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Bijections

Def: A function $f: A \to B$ is a bijection (also say "f is bijective") if it is both a surjection and an injection.

Examples

Ex: $\left.\begin{aligned} f: \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}\right]$ ↗ short for $f(x) = x^3$.

- **f is a surjection**

  - $\mathrm{Rng}(f) = \mathbb{R}$ ($\subseteq$ ✓, $\supseteq$)
  - i.e., every $y \in \mathbb{R}$ is equal to $f(x)$ for some $x \in \mathbb{R}$.

Proof: Let $y \in \mathbb{R}$.

Set $x = \sqrt[3]{y} \in \mathbb{R}$. Then $f(x) = f(\sqrt[3]{y})$

$$= (\sqrt[3]{y})^3 = y.$$

So $y \in \mathrm{Rng}(f)$, showing f is surjective. ∎

- **f is injective.**

  - $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

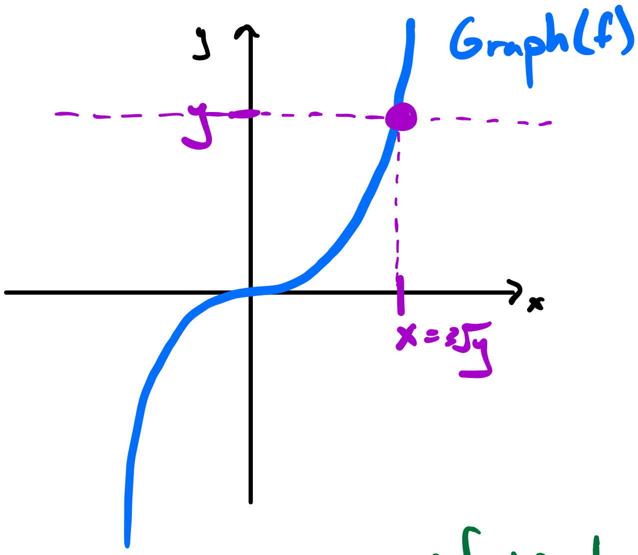
Proof: Let $x_1, x_2 \in \mathbb{R}$ and suppose $f(x_1) = f(x_2)$.

That is $x_1^3 = x_2^3$. Then $\sqrt[3]{x_1^3} = \sqrt[3]{x_2^3}$

$$x_1 = x_2.$$

∎

Picture:



- f surjective
  $\Leftrightarrow$ every $y \in \mathbb{R}$ is in the range, i.e. $y = f(x)$ for some x
  $\Leftrightarrow$ every horizontal line intersects graph.

- f injective
  $\Leftrightarrow$ $\underbrace{f(x_1) = f(x_2)}_{=y} \Rightarrow x_1 = x_2$
  $\Leftrightarrow$ each horizontal line intersects the graph at most once

So $$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3$$ is a bijection.

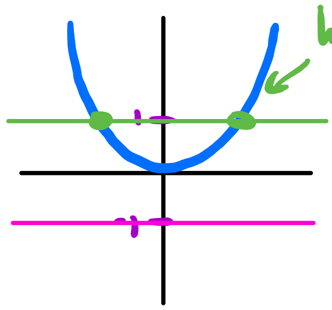
Ex: $g: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$

- g is not surjective

$\mathrm{Rng}(g) \subsetneq \mathbb{R}$
i.e. some $y \in \mathbb{R}$ is not in the range.

Proof: Let $y = -1 \in \mathbb{R}$.
Then for all $x \in \mathbb{R}$,
$g(x) = x^2 \neq -1$. So $-1 \notin \mathrm{Rng}(g)$.

Graph:



horizontal line intersects more than once

horizontal line does not intersect the graph.

- g is not injective

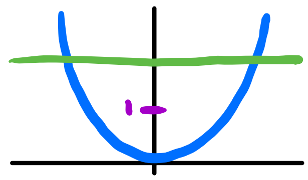
$\neg(\forall x_1, x_2)\big(\underset{T}{f(x_1) = f(x_2)} \Rightarrow \underset{F}{x_1 = x_2}\big)$
$\equiv (\exists x_1, x_2 \in \mathbb{R})\big(f(x_1) = f(x_2) \text{ but } x_1 \neq x_2\big)$

Proof: Consider $1, -1 \in \mathbb{R}$.
$g(1) = 1^2 = 1 = (-1)^2 = g(-1)$, but $1 \neq -1$.
So g is not injective. ∎

How can we make g surjective?

Easy fix: Change codomain to be range.

$$g: \mathbb{R} \to [0,\infty)$$
$$x \mapsto x^2$$

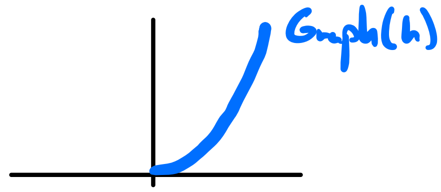


every $y \in [0,\infty)$ corresponds to a horiz. line intersects graph

still not injective.

To make it injective, restrict domain

$$h: [0,\infty) \to [0,\infty)$$
$$x \mapsto x^2$$



Summary $\quad f: A \to B$

- f surjective $\iff (\forall y \in B)(\exists x \in A)(f(x) = y)$

  f not surjective $\iff (\exists y \in B)(\forall x \in A)(f(x) \neq y)$

- f injective $\iff (\forall x_1, x_2 \in A)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$

  $\iff (\forall x_1, x_2 \in A)[x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)]$

  f not injective $\iff (\exists x_1, x_2 \in A)[x_1 \neq x_2$ and $f(x_1) = f(x_2)]$.

Together,

Lemma: Let $f: A \to B$ be a function. Then f is a bijection if and only if for every $y \in B$, there exists a unique $x \in A$ such that $f(x) = y$.

i.e. f is a bijection $\iff (\forall y \in B)(\exists! x \in A)(f(x) = y)$

- existence is surjectivity
- uniqueness is injectivity