Homework 16

Math 3345 – Spring 2022 – Kutler

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then $\gcd(a, bc) = 1$.
2. Let $a, b \in \mathbb{Z}$ and set $d = \gcd(a, b)$.
   (a) Explain why $\frac{a}{d}$ and $\frac{b}{d}$ are integers.
   (b) Prove that $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.
3. [**Falkner Section 4 Exercise 20 (The rational roots theorem)**] Let $r \in \mathbb{Q}$ be a rational number such that

   $$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0,$$

   where $n \in \mathbb{N}$ and $c_0, c_1, \ldots, c_n, c_{n-1} \in \mathbb{Z}$, and $c_n \neq 0$. In other words, r is a rational root of the polynomial $f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$, and every coefficient of this polynomial is an integer.

   Prove that r can be written in the form $r = \frac{a}{b}$. where a is an integer such that $a | c_0$ and b is a nonzero integer such that $b | c_n$.

   [Hint: Write $r = \frac{a}{b}$ as a fraction in lowest terms. Then $\gcd(a, b) = 1$. (Why?) What does this say about the prime factorizations of a and b?]

Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. [**Falkner Section 4 Exercise 21**] Let $f(x) = 3x^3 - 40x^2 + 97x + 10$.
   (a) Find a rational number $r \in \mathbb{Q}$ such that $f(r) = 0$. [Hint: Use the rational roots theorem to narrow down the possibilities for r.]
   (b) Find two other real numbers s and t such that $f(s) = 0$ and $f(t) = 0$. [Hint: Use part (a) and polynomial long division to write $f(x) = (x - r)\, g(x)$, where $g(x)$ is a quadratic polynomial.]
   (c) Explain why s and t must be irrational. [Hint: There are several ways to do this. One elegant way is to notice that $g(x) = 3h(x)$, where $h(x)$ is a quadratic polynomial to which it is particularly easy to apply the rational roots theorem.]