

HOMEWORK 7  
MATH 3345 – SPRING 2022 – KUTLER

Please complete the following problems on your own paper. Solutions should be written clearly, legibly, and with appropriate style.

1. **[Falkner Section 3 Exercise 14]** For each of the following sentences, write out what it means in words, state whether it is true or false, and prove your answer.

- (a)  $(\exists! x \in \mathbb{R})(2x + 7 = 3)$ .
- (b)  $(\exists! x \in \mathbb{R})(x^2 - 4x + 3 < 0)$ .
- (c)  $(\exists! x \in \mathbb{Z})(x^2 - 4x + 3 < 0)$ .
- (d)  $(\exists! x \in \mathbb{R})(x^2 - 4x + 4 = 0)$ .
- (e)  $(\exists! x \in \mathbb{R})(x^2 - 4x + 5 = 0)$ .
- (f)  $(\forall x \in \mathbb{R})(\exists! y \in \mathbb{R})(x + y = 0)$ .
- (g)  $(\forall x \in \mathbb{R})(\exists! y \in \mathbb{R})(xy = 1)$ .
- (h)  $(\forall x \in \mathbb{R})[\text{if } x \neq 0, \text{ then } (\exists! y \in \mathbb{R})(xy = 1)]$ .
- (i)  $(\forall x \in \mathbb{R})(\exists! y \in \mathbb{R})(xy = 0)$ .
- (j)  $(\forall x \in \mathbb{R})[\text{if } x \neq 0, \text{ then } (\exists! y \in \mathbb{R})(xy = 0)]$ .

2. **[Falkner Section 5 Exercise 2]** Prove by induction that for each  $n \in \mathbb{N}$ ,

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. In class, we saw a **correct** proof of the following statement: For each  $n \in \mathbb{N}$ ,

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

Consider the **incorrect** proof of the same statement given on the next page.

*Proof. Base Case:* Since  $1 = \frac{1 \cdot 2}{2}$ ,  $P(1)$  is true.

**Inductive Step:** Let  $n \in \mathbb{N}$ . Then

$$\begin{aligned}1 + 2 + \cdots + n + (n + 1) &= \frac{(n + 1)(n + 2)}{2} \\ \frac{n(n + 1)}{2} + (n + 1) &= \frac{(n + 1)(n + 2)}{2} \\ n + 1 &= \frac{(n + 1)(n + 2)}{2} - \frac{n(n + 1)}{2} \\ n + 1 &= \left(\frac{n + 1}{2}\right)((n + 2) - n) \\ n + 1 &= n + 1.\end{aligned}$$

Thus,  $P(n + 1)$  is true. □

Explain, in complete sentences, what is wrong with this proof. How would you fix it?

### Practice Problems

It is strongly recommended that you complete the following problems. There is no need to write up polished, final versions of your solutions (although you may find this a useful exercise). Please do not submit any work for these problems.

1. **[Falkner Section 3 Exercise 8]** Which of the variables  $x$  and  $y$  is free in the sentence  $P(x, y)$ . Answer the same question about each of the four sentences  $(\exists y)P(x, y)$ ,  $(\forall x)(\exists y)P(x, y)$ ,  $(\forall x)P(x, y)$ , and  $(\exists y)(\forall x)P(x, y)$ .
2. Read the (incorrect) proof given in **[Falkner Section 5 Exercise 8]**, which purports to prove by induction that all horses have the same color.
  - (a) Modify this proof to obtain (incorrect) proofs of the following (false) statements:
    - i. All students at OSU have the same favorite food.
    - ii. All songs are in the same key.
    - iii. All rivers flow in the same direction.
    - iv. All mountains have the same height.
  - (b) What is the error in all of these proofs?