The Principle of Mathematical Induction

Let $P(n)$ be a statement about $n \in \mathbb{N}$

Ex: $P(n) =$ " $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ "

The PoMI is: Suppose

① $P(1)$ is true (base case)

and

② For any $n \in \mathbb{N}$, if $P(n)$ is true then $P(n+1)$ is true (inductive step)

then $P(n)$ is true for every $n \in \mathbb{N}$.

In symbols,

$$\Big\{P(1) \wedge \Big[(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))\Big]\Big\}$$

$$\Rightarrow (\forall n \in \mathbb{N})\, P(n).$$

The PoMI is an <u>axiom</u> of the natural numbers. We assume it to be true.

Why should we accept it? Dominoes, trains, etc.

<u>Thm</u>: For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

<u>Proof</u>: Let $P(n) =$ "$1+2+\cdots+n = \frac{n(n+1)}{2}$". We will prove $(\forall n \in \mathbb{N})P(n)$ by induction on n.

<u>Base Case</u>: When $n=1$,

$$\sum_{i=1}^{1} i = 1 \quad \text{and} \quad \frac{1[(1)+1]}{2} = 1$$

so $P(1) =$ "$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$" is true.

<u>Inductive Step</u>: Let $n \in \mathbb{N}$. We will show

$$P(n) \Rightarrow P(n+1).$$

Suppose $P(n)$ is true. Then

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

i.e. $$1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

Add $n+1$ to both sides to get

$$\begin{aligned} 1+2+3+\cdots+n+(n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)[(n+1)+1]}{2}. \end{aligned}$$

Therefore, $P(n+1)$ is true.

We conclude, by the principle of mathematical induction, that $P(n)$ is true for all $n \in \mathbb{N}$. ∎

Thm: For every $n \in \mathbb{N}$,

$$\underbrace{1 + 3 + 5 + \cdots + (2n-1)}_{= \sum_{i=1}^{n} (2i-1)} = n^2.$$

Proof: We proceed by induction on n.

Let $P(n)$ be "$1+3+5+\cdots+(2n-1) = n^2$."

Base Case: When $n=1$, $P(1)$ is

"$1 = 1^2$"

which is true.

Inductive Step: Let $n \in \mathbb{N}$. We wish to prove $P(n) \Rightarrow P(n+1)$, so we may assume $P(n)$.

Thus,

$$1+3+5+\cdots+(2n-1) = n^2$$

is true (for <u>this</u> n).

Now,

$$1 + 3 + 5 + \cdots + (2n-1) + [2(n+1) - 1]$$
$$= n^2 + [2n + 2 - 1]$$
$$= n^2 + 2n + 1$$
$$= (n+1)^2.$$

Thus, we have shown that $P(n+1)$ is true, completing the inductive step.

By induction, we conclude that $P(n)$ is true for all $n \in \mathbb{N}$. ■

Does the base case have to be $n=1$?

No!

Ex: For every $n \in \mathbb{N}$ with $n > 3$,

$$2^n < n!$$

Check: $2^4 = 16 < 24 = 4!$
$2^5 = 32 < 120 = 5!$
$2^6 = 64 < 720 = 6!$

Proof: Let $P(n) =$ "$2^n < n!$" We will show $P(n)$ is true for all $n \in \mathbb{N}$ with $n > 3$ by induction.

Base case: When $n = 4$, we have

$$2^4 = 16 \quad \text{and} \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24,$$

so $P(4)$ is true, because $16 < 24$.

Inductive step: Let $n \in \mathbb{N}$ with $n > 3$.
We must show $P(n) \Rightarrow P(n+1)$.

Assume $P(n)$ is true, so $2^n < n!$.

Multiply both sides by 2. Since $2 < 3 < n < n+1$, we have

$$2 \cdot 2^n < 2 \cdot n! < (n+1)n!$$

i.e.

$$2^{n+1} < (n+1)!$$

Thus, $P(n+1)$ is true.

We conclude that $P(n)$ is true for all $n \in \mathbb{N}$ with $n > 3$. ∎

<u>Note</u>: We could have set

$$Q(n) = P(n+3) = \text{"} 2^{n+3} < (n+3)! \text{"}$$

and proved $(\forall n \in \mathbb{N})\, Q(n)$ by induction starting at $n = 1$.