Warm-Up: Use induction to show

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$$

for every non-negative integer n.

Parity

Def: Let $n \in \mathbb{Z}$ be an integer.

(a) n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$.

(b) n is **odd** if $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$

These are $\exists$ statements!

Thm: Every integer is even or odd.

Proof: We first show that every natural number is even or odd.
We do this by induction.

Let $P(n) =$ "n is even or odd"

Base Case: When $n=1$, we have

$$1 = 2(0) + 1$$

so n is odd. Thus $P(1)$ is true.

Inductive Step: Let $n \in \mathbb{N}$ and suppose $P(n)$ is true. Then n is even or odd.

Case 1: If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$. But then

$$n+1 = 2k+1$$

is odd.

Case 2: If n is odd, then $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$.
Now,

$$n+1 = (2\ell+1)+1 = 2\ell + 2 = 2(\ell+1)$$

so $n+1$ is even.

Thus, in both cases $P(n+1)$ holds true.

We conclude that $P(n)$ is true for every $n \in \mathbb{N}$, i.e. each $n \in \mathbb{N}$ is even or odd.

What about the remaining integers?

Zero: $0 = 2(0)$, so 0 is even.

Negatives: Suppose n is a negative integer. Then $-n \in \mathbb{N}$, so by above either $-n$ is even or $-n$ is odd.

Case 1: If $-n$ is even, then $-n = 2k$ for some $k \in \mathbb{Z}$. Thus, $n = -2k = 2(-k)$ is also even.

Case 2: If $-n$ is odd, then $-n = 2l + 1$ for some $l \in \mathbb{Z}$.

Thus,

$$\begin{aligned} n &= -(2l+1) \\ &= -2l - 1 \\ &= -2l - 2 + 1 \\ &= -2(l-1) + 1 \end{aligned}$$

is odd.

$\blacksquare$

Is there any integer which is both even and odd?

This would imply 1 is even!

Or, equivalently, $\frac{1}{2} \in \mathbb{Z}$.

How do we know this can't be so?