Recall: $\mathbb{N}$ = set of positive integers, as defined by axioms 7–10.

Def: For $a, b \in \mathbb{Z}$, $a < b$ means $b - a \in \mathbb{N}$.

- $a \leq b$ means $a < b$ or $a = b$.
- $a > b$ means $b < a$.

Ex: $0 < a \iff a - 0 \in \mathbb{N} \iff a \in \mathbb{N}$.

Lemma: Let $a, b, c \in \mathbb{Z}$.

① If $a < b$, then $a + c < a + b$
② If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$.

Proof: ① Suppose $a < b$. Then $b - a \in \mathbb{N}$.
Since

$$(b+c) - (a+c) = b - a,$$

we have $(b+c) - (a+c) \in \mathbb{N}$.
That is,

$$a + c < b + c.$$

② Suppose $a<b$ and $c>0$. Then $b-a \in \mathbb{N}$ and $c \in \mathbb{N}$.
By Positive Closure, $(b-a)\cdot c \in \mathbb{N}$.
By the Distributive Law,

$$b\cdot c - a\cdot c \in \mathbb{N},$$

i.e. $a\cdot c < b\cdot c$. ☒

The Well-Ordering axiom (#10) is the only one we haven't used yet. It says that any non-empty subset of $\mathbb{N}$ has a <u>smallest element</u>.

An element $a \in S$ is the <u>smallest element</u> of S if for all $x \in S$, $a \leq x$.

In symbols: $(\forall x \in S)(a \leq x)$

Observe that a smallest element in S, if it exists, is unique.

$(\forall x \in S)(a \leq x)$ and $(\forall x \in S)(b \leq x)$
implies $a \leq b$ and $b \leq a$, so $a = b$.

Lemma: The integer 1 is the smallest element of $\mathbb{N}$.

Proof: First, we know $\mathbb{N}$ has a smallest element by the Well-Ordering axiom. Call it a.

Since $a \leq n$ for every $n \in \mathbb{N}$, we have $a \leq 1$. Therefore $a = 1$ or $a < 1$. If $a = 1$, we are done.

To show $a < 1$ is false, we will assume it's true and derive a contradiction.

Formally, if $P \Rightarrow$ (a false statement) is true, then it must be that P is false.

So assume $a < 1$. Because $a \in \mathbb{N}$, $0 < a$.

Multiply the inequality $a < 1$ by $a > 0$ to get

$$a \cdot a < 1 \cdot a = a.$$

Now, $a \cdot a \in \mathbb{N}$ by Positive Closure, which contradicts a being the smallest element of $\mathbb{N}$.

Thus, our assumption that $a<1$ is false, so $a=1$. ■

This actually shows that the integers are what you think they are:

- We know $0, 1 \in \mathbb{Z}$ by the Identity axiom.
- We also know

$$\begin{aligned} 1+1 &= 2 \\ 2+1 &= 3 \\ 3+1 &= 4 \\ &\vdots \end{aligned}$$

are positive integers.

If there was another positive number x not on this list, then it would satisfy $n < x < n+1$ for some $n \in \mathbb{N}$. But then $0 < x-n < 1$, which violates the theorem.

- The only other integers, by Trichotomy, satisfy $-a \in \mathbb{N}$. That is, they are the additive inverses of elements in $\mathbb{N}$.

Together, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

<u>Note</u>: The handout shows this slightly more rigorously, by proving the Principle of Mathematical Induction.

Back to even and odd numbers.

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<u>Thm</u>: Let $x, y \in \mathbb{Z}$.

(a) If x is odd and y is odd, then $x+y$ is even.

(b) If x is even and y is even, then $x+y$ is even.

(c) If x is even and y is odd, then $x+y$ is odd.

Proof: (a) Suppose $x, y \in \mathbb{Z}$ are both odd numbers. Then there exist integers k_1 and k_2 such that

$$x = 2k_1 + 1$$

and

$$y = 2k_2 + 1.$$

Then

$$\begin{aligned} x + y &= (2k_1 + 1) + (2k_2 + 1) \\ &= 2k_1 + 2k_2 + 2 \\ &= 2(k_1 + k_2 + 1). \end{aligned}$$

That is, $x+y$ is even.

(b), (c) exercises.

$\blacksquare$

Thm: Let $x, y \in \mathbb{Z}$.

(a) If x is even or y is even, then $x \cdot y$ is even.

(b) If x is odd and y is odd, then $x \cdot y$ is odd.

Proof: (a) Suppose x is even or y is even.

Case 1: x is even.
Then there is $k \in \mathbb{Z}$ such that $x = 2k$. Thus,

$$x \cdot y = (2k) \cdot y = 2 \cdot (k \cdot y)$$

is even.

Case 2: y is even.
Then there is $l \in \mathbb{Z}$ such that $y = 2l$. Thus,

$$x \cdot y = x \cdot (2l) = 2 \cdot (x l)$$

is even.

(b) exercise. ∎