Warm-Up: Prove or disprove:

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers in lowest terms, then

$$\frac{ad+bc}{bd}$$

is also in lowest terms.

Irrational Numbers

Def: Let $x \in \mathbb{R}$. We say x is <u>irrational</u> if $x \notin \mathbb{Q}$.

That is, for all $a, b \in \mathbb{Z}$ with $b \neq 0$, $x \neq \frac{a}{b}$.

Ex:
- $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$ are irrational
- $\sqrt{n}$ is irrational if $n \in \mathbb{Z}$ is not a perfect square.
- $\sqrt[3]{2}$ is irrational
- π and e are irrational

Hard to prove any of these!
- π Lambert, 1761
- e Euler, 1731

To show x is irrational, we assume it is rational and get a contradiction.

Thm: Let $x \in \mathbb{Q}$ and let $y \in \mathbb{R}$ be irrational.

① $x+y$ is irrational.

② If $x \neq 0$, then $x \cdot y$ is irrational

Proof: ① Suppose, to get a contradiction, that $x+y \in \mathbb{Q}$. Since x is rational, $-x$ is rational (HW 12). Thus,

$$y = (x+y) + (-x)$$

is the sum of two rational numbers, so $y \in \mathbb{Q}$, a contradiction.

② HW 13.

What about the sum of two irrational numbers?

- <u>It can be rational</u>: $\sqrt{2}$ is irrational.
  So is $-\sqrt{2} = (-1) \cdot \sqrt{2}$.
  But $\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$.

- <u>It can be irrational</u>: $\sqrt{2} + \sqrt{2} = \underset{\text{rational}}{2}\underset{\text{irrational}}{\sqrt{2}}$ is irrational

The same thing happens with multiplication:

$$\underset{\text{irr.}}{\sqrt{2}} \cdot \underset{\text{irr.}}{\sqrt{2}} = 2 \in \mathbb{Q} \qquad \underset{\text{irr.}}{\sqrt{2}} \cdot \underset{\text{irr.}}{\sqrt{3}} = \sqrt{6} \notin \mathbb{Q}$$

Let's prove that $\sqrt{2}$ is irrational. We'll use

Fact: Let $n \in \mathbb{Z}$. If n^2 is even, then n is even.
(HW 9)

Thm: For every $x \in \mathbb{Q}$, $x^2 \neq 2$.

This actually only shows $\sqrt{2} \notin \mathbb{Q}$. To prove that $\sqrt{2}$ is a real number, you need to use the Least Upper Bound Property.

Proof: Suppose, to get a contradiction, that there is some $x \in \mathbb{Q}$ such that $x^2 = 2$.

Let $x = \frac{a}{b}$ be the representation of x in lowest terms, where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.

This means: If $d \in \mathbb{N}$ and $d \mid a$ and $d \mid b$, then $d = 1$.

We have $x^2 = \left(\frac{a}{b}\right)^2 = 2$, so $\frac{a^2}{b^2} = 2$. Therefore,

$$a^2 = 2b^2. \qquad (*)$$

Since $b^2 \in \mathbb{Z}$, this shows a^2 is even, and thus a is even as well.

Then $a = 2k$ for some $k \in \mathbb{Z}$. Now $(*)$ becomes

$$(2k)^2 = 2b^2$$
$$4k^2 = 2b^2.$$

We may divide both sides by 2 (or use Multiplicative Cancellation) to get

$$2k^2 = b^2.$$

But this means b^2 is even, and thus so is b.

Now $2 | a$ and $2 | b$, which contradicts $x = \frac{a}{b}$ being in lowest terms.

We conclude that there is no such x in $\mathbb{Q}$. $\blacksquare$

A stronger statement

Book (Example 4.52): If $x \in \mathbb{Q}$ and $x^2 \in \mathbb{Z}$, then $x \in \mathbb{Z}$.

Proof uses a similar idea: Write x in lowest terms, and see that the denominator must be 1

Key fact: If p is prime and $x, y \in \mathbb{Z}$, then

$$p \mid xy \implies p \mid x \text{ or } p \mid y.$$

We'll prove this soon.

It follows that for every $n \in \mathbb{N}$, either

- $\sqrt{n} \in \mathbb{N}$, i.e. n is a perfect square

or

- $\sqrt{n} \notin \mathbb{Q}$, i.e. $\sqrt{n}$ is irrational.