**Warm-Up**: List all subsets of

- $\{1\}$
- $\{1,2\}$
- $\{1,2,3\}$

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**Ex**: Let $n \in \mathbb{N}$. A set with $n$ elements has exactly $2^n$ subsets. Why?

**Def**: Let $A$ and $B$ be sets. We say $A$ is a <u>proper subset</u> of $B$, written $A \subsetneq B$, if $A \subseteq B$ and $A = B$.

So $A \subsetneq B$ means

$$(\forall x)(x \in A \Rightarrow x \in B) \wedge (\exists y)(y \in B \wedge y \notin A).$$

**Warning**: Some people use $\subset$ instead of $\subseteq$. $\subset$ does <u>not</u> mean proper subset.

**Warning:** $\in$ vs. $\subseteq$

**Ex:**

$1 \in \{1,2,3\}$ is true

$\{1\} \in \{1,2,3\}$ is false

$\{1\} \subseteq \{1,2,3\}$ is true

$1 \subseteq \{1,2,3\}$ makes no sense

**Ex:** $\varnothing \subseteq \varnothing$ (because $\varnothing \subseteq A$ for every set $A$)

but $\varnothing \notin \varnothing$ (because $x \in \varnothing$ is always false)

**Ex:** Consider $\{\varnothing\}$, the set whose only element is $\varnothing$.

Then $\varnothing \in \{\varnothing\}$ and $\varnothing \subseteq \{\varnothing\}$.

**Thm:** ① For all sets $A$, $A \subseteq A$. [Reflexive]

② For all sets $A$ and $B$, if $A \subseteq B$ and $B \subseteq A$, then $A = B$. [Antisymmetric]

③ For all sets $A$, $B$, and $C$, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. [Transitive]

**Note:** $\leq$ and divisibility have these same 3 properties!

<u>Proof</u>: ① we proved last time.

② is our definition of set equality.

③: Suppose $A \subseteq B$ and $B \subseteq C$. This means

$$x \in A \Rightarrow x \in B \quad \text{is true for every } x$$

and

$$x \in B \Rightarrow x \in C \quad \text{is true for every } x.$$

To prove $A \subseteq C$, suppose $x \in A$ for some $x$. Then $x \in B$ because $A \subseteq B$. Thus, $x \in C$ because $B \subseteq C$.

Therefore, $x \in A \Rightarrow x \in C$ for every $x$, so $A \subseteq C$.

■

# Algebra of Sets

**Def:** Let $A$ and $B$ be sets.

① The <u>union</u> of $A$ and $B$ is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

② The <u>intersection</u> of $A$ and $B$ is the set

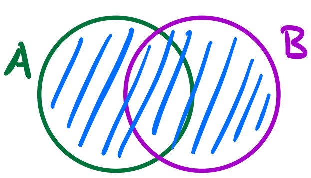
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

③ The <u>relative complement</u> of $B$ in $A$ is the set

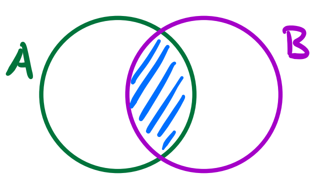
$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Also called <u>set difference</u>

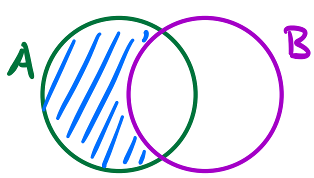
**<u>Pictures</u>:**




$A \cup B$




$A \cap B$




$A \setminus B$

**Ex**: Let $E = \{n \in \mathbb{N} \mid n \text{ is even}\}$
$= \{2, 4, 6, 8, \dots\}$

and

$P = \{p \in \mathbb{N} \mid p \text{ is prime}\}$
$= \{2, 3, 5, 7, 11, \dots\}$

- $E \cup P = \{n \in \mathbb{N} \mid n \text{ is even or prime}\}$
  $= \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 17, \dots\}$

- $E \cap P = \{n \in \mathbb{N} \mid n \text{ is even and prime}\}$
  $= \{2\}$

- $E \setminus P = \{4, 6, 8, 10, \dots\}$

- $P \setminus E = \{3, 5, 7, 11, \dots\}$

- $\mathbb{N} \setminus E = \{n \in \mathbb{N} \mid n \text{ is odd}\}$
  $= \{1, 3, 5, 7, \dots\}$

- $E \setminus \mathbb{N} = \emptyset$