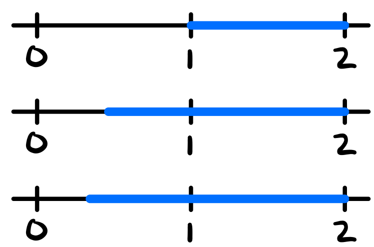
<u>Warm-Up</u>: Let $\mathcal{A} = \{ \underbrace{[0,n]}_{=\{x \in \mathbb{R} \mid 0 \leq x \leq n\}} \mid n \in \mathbb{N} \}$.

Find $\bigcup_{A \in \mathcal{A}} A$ and $\bigcap_{A \in \mathcal{A}} A$.

<u>Ex</u>: Let $A_n = [\frac{1}{n}, 2]$ for each $n \in \mathbb{N}$.

$A_1 = [1,2]$
$A_2 = [\frac{1}{2}, 2]$
$A_3 = [\frac{1}{3}, 2]$
$\vdots$



Then $\bigcap_{i=1}^{\infty} A_n = [1,2]$.

<u>Proof</u>: Left to you.

And $\bigcup_{i=1}^{\infty} A_n = (0,2]$.

Proof: Each $A_n \subseteq (0,2]$, so $\bigcup_{n=1}^{\infty} A_n \subseteq (0,2]$

why?

Now, let $x \in (0,2]$.
By the Archimedean Property (Bonus Problem #5), there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < x$.

Thus, $x \in A_m = [\frac{1}{m}, 2]$, and so $x \in \bigcup_{n=1}^{\infty} A_n$.
That is, $(0,2] \subseteq \bigcup_{n=1}^{\infty} A_n$.

∎

Ex: Similarly, if $B_n = (-\frac{1}{n}, 2]$, then

$$\bigcup_{n=1}^{\infty} B_n = (-1,2] \quad \text{and} \quad \bigcap_{n=1}^{\infty} B_n = [0,2].$$

Thm: Let $\mathcal{A}$ be a non-empty set of sets. Let $A_0 \in \mathcal{A}$. Then

$$\bigcap_{A\in\mathcal{A}} A \underset{①}{\subseteq} A_0 \underset{②}{\subseteq} \bigcup_{A\in\mathcal{A}} A.$$

Proof: ① Let $x \in \bigcap_{A\in\mathcal{A}} A$. Then for all $A \in \mathcal{A}$, $x \in A$. In particular, $x \in A_0$. Thus, $\bigcap_{A\in\mathcal{A}} A \subseteq A_0$.

② Let $x \in A_0$. Then there exists some $A \in \mathcal{A}$ such that $x \in A$, because we could take $A = A_0$. This means $x \in \bigcup_{A\in\mathcal{A}} A$. Therefore, $A_0 \subseteq \bigcup_{A\in\mathcal{A}} A$. ∎

Thm (Generalized DeMorgan Laws for sets):
Let S be a set and let $\mathcal{A}$ be a set of sets. Then

(i) $S \setminus \left(\bigcup_{A\in\mathcal{A}} A \right) = \bigcap_{A\in\mathcal{A}} (S \setminus A)$

(ii) $S \setminus \left(\bigcap_{A\in\mathcal{A}} A \right) = \bigcup_{A\in\mathcal{A}} (S \setminus A)$.

Thm (Generalized Distributive Laws for sets):
Let S be a set and let $\mathcal{A}$ be a set of sets.
Then

(i) $S \cap \left(\bigcup_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} (S \cap A)$

(ii) $S \cup \left(\bigcap_{A \in \mathcal{A}} A \right) = \bigcap_{A \in \mathcal{A}} (S \cup A)$.

The Power Set

Def: Let A be a set. The power set of A, denoted $\mathcal{P}(A)$ is the set of all subsets of A.

$$\mathcal{P}(A) = \{ S \mid S \subseteq A \}.$$

Ex: $A = \{1,2\}$. Then $\mathcal{P}(A) = \{ \emptyset, \{1\}, \{2\}, \{1,2\} \}$.

If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.

Cartesian Products

"Def": An ordered pair is a list of two objects in order.

If a and b are objects, then (a,b) denotes the ordered pair with first entry a and second entry b.

What do we mean by "in order"?

Fundamental Property: $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.

Ex:
- If $a \neq b$, then $(a,b) \neq (b,a)$.
- For any a, (a,a) is a perfectly fine ordered pair.

Compare with sets:
- $\{a,b\} = \{b,a\}$
- $\{a,a\} = \{a\}$

Aside: There is an "implementation" of ordered pairs as sets. To do this, define

$$(a,b) = \{\{a\},\{a,b\}\}.$$

Then you can prove that $(a,b) = (c,d) \iff a=c$ and $b=d$.