Cartesian Products

Def: Let A and B be sets. The Cartesian product of A and B is the set

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}.$$

Ex: Let $A = \{a, b, c\}$ and $B = \{2, 4\}$. Then

$$A \times B = \{(a,2), (a,4), (b,2), (b,4), (c,2), (c,4)\}.$$

We write $A^2 = A \times A$.

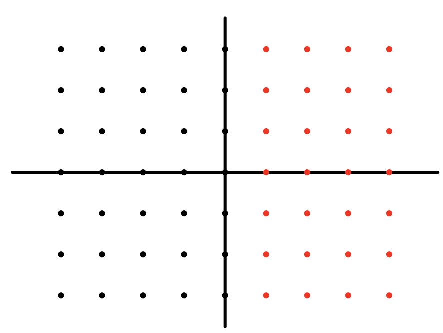
Ex: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$ is the usual Cartesian plane.

Ex:

$$\mathbb{N} \times \mathbb{Z} = \{(m,n) \mid m \in \mathbb{N},\ n \in \mathbb{Z}\}.$$

$$\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{(m,n) \mid m \in \mathbb{Z},\ n \in \mathbb{Z}\}.$$

Picture:



Note that $\mathbb{N} \times \mathbb{Z} \subseteq \mathbb{Z}^2 \subseteq \mathbb{R}^2$.

For sets A, B, C, we can similarly define

$$A \times B \times C = \{ (a,b,c) \mid a \in A,\ b \in B,\ c \in C \}.$$

ordered triples

More generally, we can define the Cartesian product of n sets to be the set of ordered <u>n-tuples</u>.

<u>Ex</u>: $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{ (x,y,z) \mid x, y, z \in \mathbb{R} \}.$

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n} = \{ (x_1, x_2, \ldots, x_n) \mid \text{each } x_i \in \mathbb{R} \}.$$

Functions

"Def": Let A and B be sets. A function $f: A \to B$ is a rule which associates to each $a \in A$ an element $f(a) \in B$.

- A is the domain of f, written $A = \mathrm{Dom}(f)$.
  (the set of all valid inputs)
  We might say f is a function on A.

- B is the target or codomain of f.
  (a set containing all possible outputs)

- For $a \in A$, $f(a)$ is the value of f at a.
  [f is the function, $f(a)$ is an element of B]

- The word map is a synonym for function.

Note that to define a function, we must specify both the domain and the target.

Ex: Let $A = \{a, b, c, d\}$. Define $f: A \to \mathbb{Z}$ by

$$f(a) = 2, \quad f(b) = 3, \quad f(c) = 1, \quad f(d) = 1.$$

When the domain is finite, like it is here, we can represent the function as a table.

Dummy variable

x	$f(x)$
a	2
b	3
c	1
d	1

Ex: Consider the functions

Dummy variable

$f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

$g: \mathbb{R} \to [0, \infty)$ " " $g(x) = x^2$

$h: \mathbb{R} \to [-2, \infty)$ " " $h(x) = x^2$

$i: [0, \infty) \to [0, \infty)$ " " $i(x) = x^2$

$j: [1, 2] \to \mathbb{R}$ " " $j(x) = x^2$

$k: [1, 2] \to [1, 4]$ " " $k(x) = x^2$

Q: Why can't we add $\ell: [1, 3] \to [1, 5]$ given by $\ell(x) = x^2$ to this list?

Def: We say two functions f and g are **equal** if

and

① $\mathrm{Dom}(f) = \mathrm{Dom}(g)$

② For every $x \in \mathrm{Dom}(f)$, $f(x) = g(x)$.

↑ equality of values

In this case we write $f = g$.

↑ equality of functions

Ex: In the previous example, we have 3 functions up to equality: $f = g = h$, i, and $j = k$.

Def: Let $f: A \to B$ be a function. The **range** of f, denoted $\mathrm{Rng}(f)$, is the set

$$\mathrm{Rng}(f) = \{ b \in B \mid f(a) = b \text{ for some } a \in A \}.$$

Note: $\mathrm{Rng}(f) \subseteq B$ automatically.

Informally, $\mathrm{Rng}(f)$ is the set of all function values.

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.
Then $\operatorname{Rng}(f) = [0, \infty)$.

Proof: ($\subseteq$) Let $y \in \operatorname{Rng}(f)$. Then $y \in \mathbb{R}$ and $y = f(x)$ for some $x \in \mathbb{R}$. Thus $y = x^2 \geq 0$, so $y \in [0, \infty)$.

($\supseteq$) Let $y \in [0, \infty)$. Then $y \geq 0$, so $\sqrt{y} \in \mathbb{R}$. Set $x \in \sqrt{y}$. We have

$$f(x) = x^2 = (\sqrt{y})^2 = y,$$

which shows that $y \in [0, \infty)$. ∎