Function Composition

Def: Let $f: A \to B$ and $g: B \to C$ be functions. The composition of g with f is the function

$$g \circ f : A \to C$$

given by

$$(g \circ f)(a) = g(f(a))$$

"g after f"

for all $a \in A$.

Ex: $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2^x$
$g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$.

Then $(g \circ f)(x) = (2^x)^2 = 2^{2x} = 4^x$

and

$$(f \circ g)(x) = 2^{(x^2)}$$

$g \circ f \neq f \circ g$, since $(g \circ f)(1) = 4$
$(f \circ g)(1) = 2$

Order matters!

Note: · Read compositions from right to left

- Sometimes, $g \circ f$ is defined but $f \circ g$ is not

Picture:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C
\end{tikzcd}
$$

Thm: Let $f: A \to B$, $g: B \to C$, and $h: C \to D$ be functions. Then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Proof idea: Both are given by $x \mapsto h(g(f(x)))$.

Surjections

Def: Let $f: A \to B$ be a function.
We say f is a surjection if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Also say: f is surjective, f is onto.

Equivalently: $f: A \to B$ is surjective $\iff \mathrm{Rng}(f) = B$

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$.
Then f is a surjection.

Proof: Let $y \in \mathbb{R}$. Set $x = \sqrt[3]{y} \in \mathbb{R}$. Then

$$f(\sqrt[3]{y}) = (\sqrt[3]{y})^3 = y. \qquad \blacksquare$$

Ex: Let $g: \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2$.
Then g is not surjective.

Proof: Consider $-1 \in \mathbb{R}$. Then for all $x \in \mathbb{R}$,
$g(x) = x^2 \neq -1$.

Note: f surjective $\Leftrightarrow (\forall y \in B)(\exists x \in A)(f(x) = y)$

f not surjective $\Leftrightarrow (\exists y \in B)(\forall x \in A)(f(x) \neq y)$.

Ex: However, $h : \mathbb{R} \to [0, \infty)$ given by $h(x) = x^2$ is surjective.

Injections

Def: A function $f : A \to B$ is an *injection* if for all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Also say: f is *injective*, f is *one-to-one*.

Ex: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is an injection.

Proof: Let $x_1, x_2 \in \mathbb{R}$ and suppose $x_1^3 = x_2^3$. Then $\sqrt[3]{x_1^3} = \sqrt[3]{x_2^3}$, i.e. $x_1 = x_2$. ∎

Ex: $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ is <u>not</u> an injection.

Proof: Consider $-1, 1 \in \mathbb{R}$. Then $-1 \neq 1$, but
$$g(-1) = (-1)^2 = 1 = (1)^2 = g(1).$$ ∎

Note: f injective $\iff (\forall x_1, x_2 \in A)[\underset{T}{f(x_1) = f(x_2)} \Rightarrow \underset{F}{x_1 = x_2}]$

$$\iff (\forall x_1, x_2 \in A)[\underset{T}{x_1 \neq x_2} \Rightarrow \underset{F}{f(x_1) \neq f(x_2)}]$$

f not injective $\iff (\exists x_1, x_2 \in A)[x_1 \neq x_2 \text{ and } f(x_1) = f(x_2)]$

Ex: However, $h: [0, \infty) \to \mathbb{R}$ given by $h(x) = x^2$ <u>is</u> injective.