Warm-Up: Which of the following functions are surjections? Which are injections?

$f: \mathbb{Z} \to \mathbb{Z}$ $n \mapsto n+1$	$g: \mathbb{N} \to \mathbb{N}$ $n \mapsto n+1$	$h: \mathbb{Z} \to \{0,1,2,3,4\}$ $n \mapsto$ the remainder when n is divided by 5

Bijections

Def: A function $f: A \to B$ is a bijection if it is both a surjection and an injection.

f surjective $\iff (\forall y \in B)(\exists x \in A)[f(x) = y]$

f injective $\iff (\forall x_1, x_2 \in A)[f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$

$\iff (\forall y \in B)(\forall x_1, x_2 \in A)[(f(x_1) = y \wedge f(x_2) = y) \Rightarrow x_1 = x_2]$

Together, we get the following:

Lemma: Let $f: A \to B$ be a function. Then f is a bijection if and only if for every $y \in B$, there exists a unique $x \in A$ such that $f(x) = y$.

Ex: $f: \mathbb{Z} \to \mathbb{Z}$. For each $y \in \mathbb{Z}$, there is a unique $x \in \mathbb{Z}$ such that $f(x) = y$, namely $x = y - 1$.
$n \longmapsto n+1$

Inverse Functions

A bijection $f: A \to B$ gives us a rule for going back to B from A. Specifically, $y \in B$ can map back to the unique $x \in A$ such that $f(x) = y$.

Def: Let $f: A \to B$ be a bijection. The <u>inverse function</u> of f is

$$f^{-1}: B \to A$$

defined as follows: For each $y \in B$, $f^{-1}(y)$ is the unique element $x \in A$ such that $f(x) = y$.

That is $f^{-1}(y) = x \iff y = f(x)$.

Ex: $f: \mathbb{R} \to (0,\infty)$ given by $f(x) = e^x$ is a bijection.

$f^{-1}: (0,\infty) \to \mathbb{R}$ is given by $f^{-1}(y) = \ln(y)$.

$$\ln(y) = x \iff y = e^x$$

Ex: $g: [0,\infty) \to [0,\infty)$, $x \longmapsto x^2$ is a bijection.

Its inverse is $g^{-1}: [0,\infty) \to [0,\infty)$, $y \longmapsto \sqrt{y}$

$$\sqrt{y} = x \iff y = x^2 \text{ and } x \geq 0$$

Ex: $\sin : \mathbb{R} \to \mathbb{R}$ is not a bijection,
but $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [1,1]$ is.

Its inverse is $\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\sin^{-1}(y) = x \iff y = \sin(x) \text{ and } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}$$

Thm: Let $f : A \to B$ be a bijection and let $f^{-1} : B \to A$ be its inverse. Then

① $f^{-1} \circ f = id_A : A \to A$

and

② $f \circ f^{-1} = id_B : B \to B$

This is essentially a rephrasing of the fundamental identity $f^{-1}(y) = x \iff f(x) = y$.

Proof: ① Let $x \in A$. We must show

$$(f^{-1} \circ f)(x) = id_A(x) = x.$$

Set $y = f(x)$. Then, by definition of f^{-1}, $f^{-1}(y) = x$. But then

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x. \quad \checkmark$$

② Let $y \in B$. We must show

$$(f \circ f^{-1})(y) = id_B(y) = y.$$

Set $x = f^{-1}(y)$. Then $f(x) = y$, so

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y.$$ ✓

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<u>Cor</u>: Let $f: A \to B$ be a bijection. Then its inverse $f^{-1}: B \to A$ is also a bijection, and $(f^{-1})^{-1} = f$.

<u>Proof</u>: Let $f: A \to B$ be a bijection.

- <u>f^{-1} is surjective</u>: Let $x \in A$.

  We must find $y \in B$ so that $f^{-1}(y) = x$.

  Set $y = f(x)$. Then, by the theorem,

  $$f^{-1}(y) = f^{-1}(f(x)) = x.$$ ✓

- <u>f^{-1} is injective</u>: Let $y_1, y_2 \in B$ such that $f^{-1}(y_1) = f^{-1}(y_2)$.
  Then
  $$f(f^{-1}(y_1)) = f(f^{-1}(y_2)),$$
  so by the theorem,
  $$y_1 = y_2.$$

- <u>$(f^{-1})^{-1} = f$</u>: By definition, for $x \in A$ and $y \in B$,
  $$(f^{-1})^{-1}(x) = y \iff x = f^{-1}(y) \iff f(x) = y.$$
  Thus, $(f^{-1})^{-1} = y$.

$\blacksquare$

The following theorems are proved using similar methods.

<u>Thm</u>: Let $f: A \to B$ and $g: B \to A$ be functions. If
$$g \circ f = id_A \quad \text{and} \quad f \circ g = id_B,$$
then f is a bijection and $g = f^{-1}$.

<u>Thm</u>: If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection also, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.