**Warm-Up:** Prove that

$$f : \mathbb{R} \setminus \{3\} \to \mathbb{R} \setminus \{1\}$$
$$x \mapsto \frac{x}{x-3}$$

is a bijection.

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# Cardinality

What does it mean for a set $A$ to have exactly $n$ elements?

**Ex:** $A = \{4, \text{red}, \$\}$ has exactly 3 elements

How do we know? We can list them:

1. 4
2. red
3. \$

This is just a bijection $f : \{1, 2, 3\} \to A$.

surjection $\Leftrightarrow$ every element in $A$ is on the list

injection $\Leftrightarrow$ no element in $A$ is on the list more than once.

**Def:** Let $A$ and $B$ be sets. We say $A$ and $B$ <u>have the same cardinality</u>, denoted $|A| = |B|$, if there exists a bijection $f: A \to B$.

Book: $A$ and $B$ are <u>equinumerous</u>, $\bar{\bar{A}} = \bar{\bar{B}}$.

This is an equivalence relation.

**Thm:** Let $A, B, C$ be sets. Then

1. $|A| = |A|$. [Reflexive]
2. If $|A| = |B|$, then $|B| = |A|$. [Symmetric]
3. If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. [Transitive]

**Proof sketch:**

1. $\mathrm{id}_A: A \to A$, $x \mapsto x$ is a bijection.
2. If $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is a bijection.
3. If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection. ∎

If $A$ is a set and $n \in \mathbb{N}$ such that $A$ and $\{1, 2, \ldots, n\}$ have the same cardinality, then we say $A$ has <u>cardinality $n$</u> (or $A$ has <u>exactly $n$ elements</u>), and write $|A| = n$.

We also write $|\emptyset| = 0$.

<u>Def</u>: A set $A$ is <u>finite</u> if either

- $A = \emptyset$ (i.e. $|A| = 0$)

or

- there exists $n \in \mathbb{N}$ such that $|A| = n$.

A set is <u>infinite</u> if it is not finite.

<u>Ex</u>: $A = \{4, \text{red}, \$\}$. $|A| = 3$

<u>Ex</u>: $B = \{a, b, c, \ldots, z\}$. $|B| = 26$

<u>Ex</u>: Is there a set $C$ such that $|C| = 3$ and $|C| = 26$? No!

**Thm**: Let $n, m \in \mathbb{N} \cup \{0\}$. If $A$ is a set such that $|A| = n$ and $|A| = m$, then $n = m$.

**Proof idea**:

$|A| = n$ means there is a bijection $f: \{1, \dots, n\} \to A$
$|A| = m$ " " " $g: \{1, \dots, m\} \to A$.

So $g^{-1} \circ f : \{1, \dots, n\} \to \{1, \dots, m\}$ is a bijection.

- If $n > m$, this can't be injective.
- If $n < m$, this can't be surjective.

**Ex**: $\mathbb{N}$ is infinite

Why? Let $n \in \mathbb{N}$ and suppose $f: \{1, \dots, n\} \to \mathbb{N}$ is a function. Set

$$m = \text{maximum of } f(1), f(2), \dots, f(n).$$

Then for all $i \in \{1, \dots, n\}$, $f(i) \leq m < m+1$.
So $f(i) \neq m+1$, showing that $f$ is not surjective.
So $f$ cannot be a bijection.

**Ex**: Similarly, $\mathbb{Q}$ and $\mathbb{R}$ are infinite.

**WARNING**: It may be tempting to write

$$|\mathbb{N}| = \infty$$

$$|\mathbb{Q}| = \infty$$

$$|\mathbb{R}| = \infty$$

We will not do this.

As we will see, $|\mathbb{N}| = |\mathbb{Q}|$, but $|\mathbb{N}| \neq |\mathbb{R}|$.