**Warm-Up**: Prove that the function

$$f: \mathbb{N} \longrightarrow \mathbb{N} \setminus \{1\}$$
$$x \longmapsto x+1$$

is a bijection.

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**Thm**: Let $S$ be a finite set and $T \subseteq S$. Then

- $T$ is finite
- $|T| \leq |S|$
- $|T| = |S|$ if and only if $T = S$.

**Proof**: Book Thms 13.30, 13.33.

**Cor**: Let $A$ and $B$ be finite sets, and let $f: A \to B$ be a function. Then

1. If $f$ is an injection, then $|A| \leq |B|$
2. If $f$ is a surjection, then $|A| \geq |B|$

**Proof:** ① Suppose $f: A \to B$ is injective. Then

$$f: A \to \mathrm{Rng}(f)$$

is a bijection. Hence, $|A| = |\mathrm{Rng}(f)|$.
But $\mathrm{Rng}(f) \subseteq B$, so $|\mathrm{Rng}(f)| \leq |B|$ by the Theorem. Together, we get $|A| \leq |B|$.

② Suppose $f: A \to B$ is surjective. Since $B$ is finite, $|B| = n$ for some $n \in \mathbb{N}$, so we can write

$$B = \{b_1, b_2, \ldots, b_n\}.$$

For each $i \in \{1, \ldots, n\}$, let $a_i \in A$ be such that $f(a_i) = b_i$.

If $i \neq j$, then $f(a_i) = b_i \neq b_j = f(a_j)$, so $a_i \neq a_j$.

Thus, $|\{a_1, \ldots, a_n\}| = n$. But $\{a_1, \ldots, a_n\} \subseteq A$, so $n \leq |A|$. Since $|B| = n$, we have $|A| \geq |B|$.

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The contrapositive of ① is the

<u>Pigeonhole Principle</u>: Let $A$ and $B$ be finite sets and $f: A \to B$ a function. If $|A| > |B|$, then $f$ is not injective.

$A$ - set of pigeons
$B$ - set of pigeonholes
$f: A \to B$ puts each pigeon in a pigeonhole

Then there is a pigeonhole containing more than one pigeon.

<u>Ex</u>: If $a_1, a_2, a_3, a_4 \in \mathbb{Z}$, then the difference $a_i - a_j$ will be divisible by 3 for some $i \neq j$.

<u>Ex</u>: Suppose $n$ people are at a party. Then there are two people who have the same number of friends at the party.

# Infinite Sets

We already saw that

$$f: \mathbb{N} \to \mathbb{N} \setminus \{1\}$$
$$x \mapsto x+1$$

is a bijection, so $|\mathbb{N}| = |\mathbb{N} \setminus \{1\}|$.

Here's another example:

**Ex:** Let $E = \{n \in \mathbb{N} \mid n \text{ is even}\} = \{2, 4, 6, 8, \ldots\}$.
Then

$$g: \mathbb{N} \to E$$
$$x \mapsto 2x$$

is a bijection. Thus, $|\mathbb{N}| = |E|$.

Proof: Let $x_1, x_2 \in \mathbb{N}$. If $f(x_1) = f(x_2)$, then $2x_1 = 2x_2$, so cancelling the 2 gives $x_1 = x_2$. Thus, $f$ is injective.

Let $y \in E$. Then $y = 2k$ for some $k \in \mathbb{N}$ (why?) Thus, $f(k) = 2k = y$. This shows that $f$ is surjective.

**Def:** A set $A$ is <u>countably infinite</u> if there exists a bijection $f: \mathbb{N} \to A$. That is, if $|A| = |\mathbb{N}|$.

A set is <u>countable</u> if it is finite or countably infinite.

A set is <u>uncountable</u> if it is not countable.

**Ex:** $\mathbb{N}$ is countably infinite
$\mathbb{N} \setminus \{1\}$ is countably infinite
$E = \{n \in \mathbb{N} \mid n \text{ is even}\}$ is countably infinite.
$\mathbb{Z}$ is countably infinite

**Think:** Countably infinite sets can be enumerated in an infinite list.

**Thm:** Let $A$ be a countably infinite set. Then any subset $B \subseteq A$ is countable.