Warm-Up: Can you find a bijection

$$f : \mathbb{N} \longrightarrow \mathbb{Z} \,?$$

Ex: $\mathbb{N} \times \mathbb{N}$ is countably infinite

Key: Write the elements of $\mathbb{N} \times \mathbb{N}$ in a grid

	1	2	3	4	5	$\cdots$
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	
$\vdots$						$\ddots$

Define a bijection $f : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ by reading along the northeast diagonals in order:

$$f(1) = (1,1)$$
$$f(2) = (2,1)$$
$$f(3) = (1,2)$$
$$f(4) = (3,1)$$
$$\vdots$$

Ex: The set $\mathbb{Q}_{>0} = \{ q \in \mathbb{Q} \mid q > 0 \}$ of positive rational numbers is countably infinite.

Key idea: Each $q \in \mathbb{Q}_{>0}$ can be written uniquely as $q = \frac{a}{b}$ where

- $a, b \in \mathbb{N}$

and

- $\frac{a}{b}$ is in lowest terms $(\gcd(a,b) = 1)$

Now, use a grid again, but cross out fractions not in lowest terms:

	1	2	3	4	5	$\cdots$
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	
2	$\frac{2}{1}$	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$	
3	$\frac{3}{1}$	$\frac{3}{2}$	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$	
4	$\frac{4}{1}$	$\cancel{\frac{4}{2}}$	$\frac{4}{3}$	$\cancel{\frac{4}{4}}$	$\frac{4}{5}$	
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\cancel{\frac{5}{5}}$	
$\vdots$						$\ddots$

Define a bijection $g : \mathbb{N} \to \mathbb{Q}_{>0}$ by reading the remaining entries along the northeast diagonals.

$$g(1) = 1, \quad g(2) = 2, \quad g(3) = \tfrac{1}{2}, \quad g(4) = 3, \quad g(5) = \tfrac{1}{3}, \quad \ldots$$

<u>Ex</u>: $\mathbb{Q}$ is countably infinite.

Let $g: \mathbb{N} \to \mathbb{Q}_{\geq 0}$ be the bijection above.
Define a bijection $h: \mathbb{N} \to \mathbb{Q}$ by

$$h(n) = \begin{cases} 0 & \text{if } n = 1 \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ -g\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

So

$$\begin{aligned} h(1) &= 0 \\ h(2) &= g(1) = 1 \\ h(3) &= -g(1) = -1 \\ h(4) &= g(2) = 2 \\ h(5) &= -g(2) = -2 \\ &\vdots \end{aligned}$$

Thm: $|\mathbb{R}| \neq |\mathbb{N}|$ ($\mathbb{R}$ is uncountable)

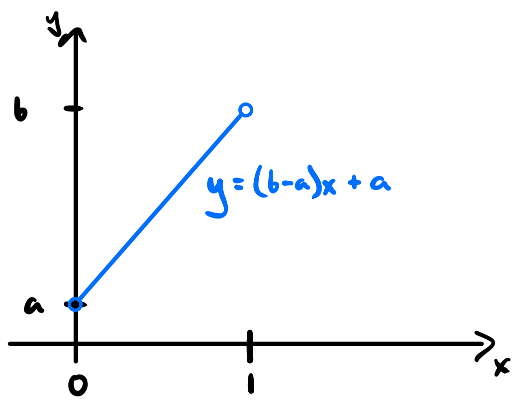
Step 1: If $a, b \in \mathbb{R}$ with $a < b$, then $|(a,b)| = |(0,1)|$.

We must give a bijection between $(0,1)$ and (a,b).

A linear function will work:

$$f : (0,1) \to (a,b)$$
$$x \mapsto (b-a)x + a$$

Graph:



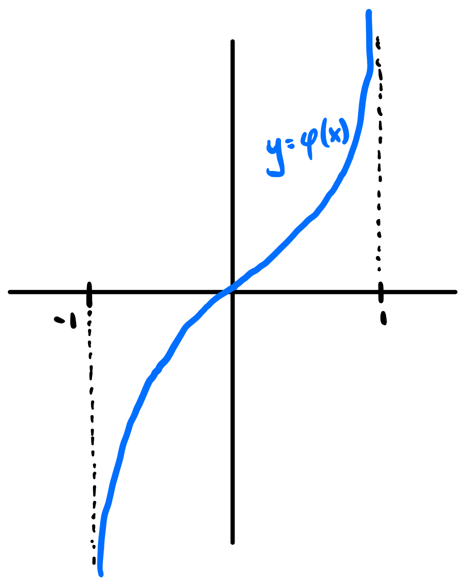
Exercise: Check that f is a bijection.

Step 2: $|\mathbb{R}| = |(-1,1)|$

There are many ways to do this, but we'll use the book's: Define

$$\varphi : (-1,1) \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{x}{1-|x|}$$

Graph:



Exercise: Check that φ is a bijection. (Follows from HW 23.)

Step 3: There is no surjection $\mathbb{N} \to (0,1)$ (and thus no bijection $\mathbb{N} \to (0,1)$).

Why is this enough? If $|\mathbb{N}| = |\mathbb{R}|$, then since $|\mathbb{R}| = |(-1,1)|$ and $|(-1,1)| = |(0,1)|$, transitivity gives $|\mathbb{N}| = |(0,1)|$, a contradiction.

To show this, we use Cantor's Diagonal Argument.

Need:
- Every real number has an infinite decimal representation.

$$\text{e.g.} \quad \tfrac{1}{3} = 0.3333333\cdots$$
$$\tfrac{3}{4} = 0.7500000\cdots$$
$$\pi - 3 = 0.14159265\cdots$$

- This representation is unique if we don't allow infinite repeating 9s.

$$\text{e.g.} \quad \tfrac{3}{4} = 0.749999999\cdots$$
$$= 0.75000000\cdots$$

Now, let $f: \mathbb{N} \to (0,1)$ be a function.
Think of this as an infinite list:

$$c_1 = f(1) = 0.x_{11}x_{12}x_{13}x_{14}x_{15}\cdots$$
$$c_2 = f(2) = 0.x_{21}x_{22}x_{23}x_{24}x_{25}\cdots$$
$$c_3 = f(3) = 0.x_{31}x_{32}x_{33}x_{34}x_{35}\cdots$$
$$c_4 = f(4) = 0.x_{41}x_{42}x_{43}x_{44}x_{45}\cdots$$
$$\vdots$$

x_{nm} = mth digit of nth number

Define a number c_0 by

$$c_0 = 0.x_{01}x_{02}x_{03}x_{04}x_{05}\cdots$$

where

$$x_{0m} = \begin{cases} 1 & \text{if } x_{mm} \neq 1 \\ 2 & \text{if } x_{mm} = 1 \end{cases}$$

Then $c_0 \in (0,1)$, but

$c_0 \neq c_1$ because $x_{01} \neq x_{11}$

$c_0 \neq c_2$ " $x_{02} \neq x_{22}$

$c_0 \neq c_3$ " $x_{03} \neq x_{33}$

$\vdots$

Thus, $c_0 \notin \mathrm{Rng}(f)$, so f is not surjective.

If time: Cantor's Generalized Diagonal Lemma

A similar argument shows that we can always find "larger" infinities.

Def: Let A and B be sets.

- We write $|A| \leq |B|$ if there exists an injection $A \to B$.

- We write $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$.

Note: This is consistent with what we know about finite sets, where $|A|$ and $|B|$ are non-negative integers.

Ex: $|\mathbb{N}| < |\mathbb{R}|$.

Thm (Cantor): Let A be any set. Then

$$|A| < |\mathcal{P}(A)|$$

Note: We've seen that if A is finite, then $|\mathcal{P}(A)| = 2^{|A|} > |A|$.

So the interesting (hard) part of this theorem is the case where A is infinite.

Proof: First, consider

$$g: A \longrightarrow \mathcal{P}(A)$$
$$x \longmapsto \{x\}$$

This is a bijection, since $\{x_1\} = \{x_2\}$ if and only if $x_1 = x_2$.

Thus $|A| \leq |\mathcal{P}(A)|$.

Next, we must show $|A| \neq |\mathcal{P}(A)|$.

Consider any function $f: A \to \mathcal{P}(A)$. So for any $x \in A$, we get a subset $f(x) \subseteq A$.

<u>Claim:</u> f is not surjective
(Thus, f cannot be a bijection.)

Consider
$$S = \{x \in A \mid x \notin f(x)\} \subseteq A.$$

Suppose that $S \in \mathrm{Rng}(f)$. Then $S \overset{!}{=} f(x_0)$ for some $x_0 \in A$.

Is $x_0 \in S$ or $x_0 \notin S$?

- If $x_0 \in S$, then by definition $x_0 \notin f(x_0) = S$, a contradiction.

- If $x_0 \notin S = f(x_0)$, then by definition of S, $x_0 \in S$, a contradiction.

Since both possibilities lead to a contradiction, it must be that $S \notin \mathrm{Rng}(f)$. Thus, f is not surjective.

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