<u>Warm-Up</u>: Use a truth table to show $P \Rightarrow Q$ is <u>not</u> logically equivalent to $Q \Rightarrow P$.

<u>Recall</u>: $Q \Rightarrow P$ is the <u>converse</u> of $P \Rightarrow Q$.

$\neg Q \Rightarrow \neg P$ is the <u>contrapositive</u> of $P \Rightarrow Q$, and

$$\neg Q \Rightarrow \neg P \equiv P \Rightarrow Q.$$

A final logical connective:

5) <u>Biconditional</u>: $\Leftrightarrow$ means "if and only if"

$P \Leftrightarrow Q$ is true exactly when P and Q have the same truth value.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Thm: $P \Leftrightarrow Q$ is logically equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Proof:

P	Q	$P \Leftrightarrow Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

$\blacksquare$

A sentence of the form $P \Leftrightarrow Q$ is called a <u>biconditional sentence</u>.

Ways to say $P \Leftrightarrow Q$:

"P if and only if Q"

"P is necessary and sufficient for Q"

"Q is necessary and sufficient for P"

"P is necessary for Q" is $Q \Rightarrow P$

"P is sufficient for Q" is $P \Rightarrow Q$

<u>Ex</u>: $x^2 = 9 \iff x = 3$ or $x = -3$

How do we know this is a true sentence?

Let

$$P = \text{"}x^2 = 9\text{"}$$
$$Q = \text{"}x = 3 \text{ or } x = -3\text{"}$$

<u>Show $P \Rightarrow Q$ is true</u>

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

<u>Case 1</u>: P is true. Then $x^2 = 9$, so

$$x^2 - 9 = 0.$$

Factoring, we get

$$(x-3)(x+3) = 0.$$

Hence,

$$x - 3 = 0 \quad \text{or} \quad x + 3 = 0.$$

Thus,

$$x = 3 \quad \text{or} \quad x = -3. \quad \checkmark$$

<u>Case 2</u>: P is false. Then $P \Rightarrow Q$ is vacuously true.

Show $Q \Rightarrow P$ is true

Case 1: Q is true. Then $x = 3$ or $x = -3$, so

$$x^2 = (3)^2 = 9 \quad \text{or} \quad x^2 = (-3)^2 = 9. \quad \checkmark$$

Case 2: Q is false. Then $Q \Rightarrow P$ is vacuously true.

Conditional Proof

In general, to show $P \Rightarrow Q$ is true, we must

① Assume P is true.

② Under this assumption, show that Q must be true also.

Why is this valid?

When P is false, $P \Rightarrow Q$ is automatically true.

This method is called <u>conditional proof</u>.

Most of our theorems are of the form $P \Rightarrow Q$, so we'll write a lot of conditional proofs.

To prove $P \Leftrightarrow Q$ is true, we need two conditional proofs: $P \Rightarrow Q$ and $Q \Rightarrow P$.

eq. $\neg Q \Rightarrow \neg P$ eq. $\neg P \Rightarrow \neg Q$